

# SEMIDEFINITE REPRESENTATIONS OF NON-COMPACT CONVEX SETS

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**ABSTRACT.** We consider the problem of the semidefinite representation of a class of non-compact basic semialgebraic sets. We introduce the conditions of pointedness and closedness at infinity of a semialgebraic set and show that under these conditions our modified hierarchies of nested theta bodies and Lasserre's relaxations converge to the closure of the convex hull of  $S$ . Moreover, if the PP-BDR property is satisfied, our theta body and Lasserre's relaxation are exact when the order is large enough; if the PP-BDR property does not hold, our hierarchies convergent uniformly to the closure of the convex hull of  $S$  restricted to every fixed ball centered at the origin. We illustrate through a set of examples that the conditions of pointedness and closedness are essential to ensure the convergence. Finally, we provide some strategies to deal with cases where the conditions of pointedness and closedness are violated.

**Key words.** Convex sets, semidefinite representation, theta bodies, sums of squares, moment matrices.

## 1. INTRODUCTION

Consider the basis semialgebraic set

$$S := \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \dots, g_m(x) \geq 0\},$$

where  $g_i(X) \in \mathbb{R}[X] := \mathbb{R}[X_1, \dots, X_n]$ ,  $i = 1, \dots, m$ . The convex hull of  $S$  is denoted by  $\mathbf{co}(S)$  and its closure is denoted by  $\mathbf{cl}(\mathbf{co}(S))$ . Characterizing  $\mathbf{cl}(\mathbf{co}(S))$  is an important issue raised in [1, 9, 8, 14]. There is a considerable amount of interesting work by many people. For instance, using the same variables appearing in  $S$ , Rostalski and Sturmfels [19] exploit projective varieties to explicitly find the polynomials that describe the boundary of  $\mathbf{co}(S)$  when  $S$  is a compact real algebraic variety; by introducing more variables, theta bodies [4] and Lasserre's relaxations [10] have been given to compute  $\mathbf{cl}(\mathbf{co}(S))$  approximately or exactly when  $S$  is a compact semialgebraic set. In this paper, we aim to extend works in [4, 10] and provide sufficient conditions such that the modified hierarchy of theta bodies and Lasserre's relaxations of non-compact semialgebraic set  $S$  can still converge to the closure of the convex hull of  $S$ .

Let  $\tilde{g}_1, \dots, \tilde{g}_m$  be homogenized polynomials of  $g_1, \dots, g_m$  respectively. We lift the cone of  $S$  to a cone of  $\tilde{S}^\circ \in \mathbb{R}^{n+1}$

$$\tilde{S}^\circ := \{\tilde{x} \in \mathbb{R}^{n+1} \mid \tilde{g}_1(\tilde{x}) \geq 0, \dots, \tilde{g}_m(\tilde{x}) \geq 0, x_0 > 0\}.$$

Let  $\tilde{X} := (X_0, X_1, \dots, X_n)$ . Denote  $\mathcal{Q}_k(\tilde{G})$  as the  $k$ -th quadratic module generated by

$$\tilde{G} := \left\{ \tilde{g}_1, \dots, \tilde{g}_m, X_0, \|\tilde{X}\|_2^2 - 1, 1 - \|\tilde{X}\|_2^2 \right\},$$

and

$$\tilde{S} := \{\tilde{x} \in \mathbb{R}^{n+1} \mid \tilde{g}_1(\tilde{x}) \geq 0, \dots, \tilde{g}_m(\tilde{x}) \geq 0, x_0 \geq 0, \|\tilde{x}\|_2^2 = 1\}.$$

Denote  $\mathbb{P}[\tilde{X}]_1 := (\mathbb{R}[\tilde{X}]_1 \setminus \mathbb{R}) \cup \{0\}$  where  $\mathbb{R}[\tilde{X}]_1$  is the set of linear polynomials in  $\mathbb{R}[\tilde{X}]$ . We construct the hierarchy of theta bodies  $\widetilde{\text{TH}}_k(\tilde{G})$

$$\widetilde{\text{TH}}_k(\tilde{G}) := \left\{x \in \mathbb{R}^n \mid \tilde{l}(1, x) \geq 0, \quad \forall \tilde{l} \in \mathcal{Q}_k(\tilde{G}) \cap \mathbb{P}[\tilde{X}]_1\right\},$$

and Lasserre's relaxations  $\Omega_k(\tilde{G})$

$$\Omega_k(\tilde{G}) := \left\{x \in \mathbb{R}^n \left| \begin{array}{l} \exists y \in \mathbb{R}^{\tilde{s}(2k)}, \text{ s.t. } \mathcal{L}_y(X_0) = 1, \\ \mathcal{L}_y(X_i) = x_i, \quad i = 1, \dots, n, \\ M_{k-1}(X_0 y) \succeq 0, \quad M_{k-1}((\|\tilde{X}\|_2^2 - 1)y) = 0, \\ M_k(y) \succeq 0, \quad M_{k-k_j}(\tilde{g}_j y) \succeq 0, \quad j = 1, \dots, m, \end{array} \right. \right\},$$

where  $\tilde{s}(k) = \binom{n+k+1}{n+1}$  and  $k_j = \lceil \deg g_j / 2 \rceil$ , for every  $k \in \mathbb{N}$ .

**Our contribution:** Consider a non-compact basic semialgebraic set  $S$ .

- Assuming that  $S$  is closed at  $\infty$  [12] and its homogenized cone  $\mathbf{co}(\mathbf{cl}(\tilde{S}^\circ))$  is closed and pointed (equivalently, it contains no lines through the origin):
  - We prove that the hierarchies of  $\widetilde{\text{TH}}_k(\tilde{G})$  and  $\Omega_k(\tilde{G})$  defined above converge to  $\mathbf{cl}(\mathbf{co}(S))$  asymptotically. If  $\mathcal{Q}_k(\tilde{G})$  is closed, then  $\widetilde{\text{TH}}_k(\tilde{G}) = \Omega_k(\tilde{G})$  for  $k \in \mathbb{N}$ .
  - If the Putinar-Prestel's Bounded Degree Representation (PP-BDR) [10] holds for  $\tilde{S}$  with order  $k'$ , then we conclude that  $\mathbf{cl}(\mathbf{co}(S)) = \widetilde{\text{TH}}_{k'}(\tilde{G}) = \Omega_{k'}(\tilde{G})$ . If PP-BDR property does not hold, then for every  $\epsilon > 0$ , we show that  $\widetilde{\text{TH}}_k(\tilde{G})$  and  $\Omega_k(\tilde{G})$  convergent uniformly to  $\mathbf{cl}(\mathbf{co}(S))$  restricted to every fixed ball centered at the origin.
- We show that the conditions of closedness and pointedness are essential to guarantee the convergence of the constructed hierarchies.
  - We observe that the condition of closedness of  $S$  at  $\infty$  depends on the generators of  $S$  and in many cases, we can force  $S$  to become closed at  $\infty$  by adding a redundant linear polynomial obtained by the property of pointedness of  $\mathbf{co}(\mathbf{cl}(\tilde{S}^\circ))$ .
  - If  $\mathbf{co}(\mathbf{cl}(\tilde{S}^\circ))$  is not pointed, then we divide  $S$  into  $2^n$  parts along each axis. If  $S$  is closed at  $\infty$  and each part satisfies PP-BDR property, we can compute the theta bodies and Lasserre's relaxations for each one and then glue them together properly.

**Structure of the paper:** We provide in Section 2 some preliminaries about convex sets and cones. We also recall some known results about theta bodies [4] and Lasserre's relaxations [10] for compact semialgebraic sets. An example is given to show that for a non-compact semialgebraic set  $S$ , the sequence defined in (2.5) or (2.7) does not converge to  $\mathbf{cl}(\mathbf{co}(S))$ . In Section 3, when  $S$  is a non-compact semialgebraic set, we provide sufficient conditions for guaranteeing the convergence of modified Lasserre's relaxations and theta bodies for computing  $\mathbf{cl}(\mathbf{co}(S))$ . Some examples are also given to illustrate our method. More discussions on these sufficient conditions are given in Section 4.

## 2. PRELIMINARIES

In this section we present some preliminaries needed in the rest of this paper.

**2.1. Convex sets and cones.** The symbol  $\mathbb{R}$  denotes the set of real numbers. For  $x \in \mathbb{R}^n$ ,  $\|x\|_2$  denotes the standard Euclidean norm of  $x$ . A subset  $C \subseteq \mathbb{R}^n$  is *convex* if for any  $u, v \in C$  and any  $\theta$  with  $0 \leq \theta \leq 1$ , we have  $\theta u + (1 - \theta)v \in C$ . For any subset  $W \subseteq \mathbb{R}^n$ , denote  $\mathbf{ri}(W)$ ,  $\mathbf{cl}(W)$  and  $\mathbf{co}(W)$  as the relative interior, closure and convex hull of  $W$ , respectively. A subset  $K \subseteq \mathbb{R}^n$  is a *cone* if it is closed under positive scalar multiplication. The *dual cone* of  $K$  is

$$K^* = \{c \in \mathbb{R}^n \mid \langle c, x \rangle \geq 0, \quad \forall x \in K\}.$$

In particular,  $(\mathbb{R}^n)^* = \mathbb{R}^n$  and  $L^* = L^\perp$  for any subspace  $L \subseteq \mathbb{R}^n$ . A cone  $K$  need not be convex, but its dual cone  $K^*$  is always convex and closed. The second dual  $K^{**}$  is the closure of the convex hull of  $K$ . Hence, if  $K$  is a closed convex cone, then  $K^{**} = K$ .

**Proposition 2.1.1.** *Let  $K_1$  and  $K_2 \subseteq \mathbb{R}^n$  be two closed convex cone, then*

$$(2.1) \quad (K_1 + K_2)^* = K_1^* \cap K_2^* \quad \text{and} \quad (K_1 \cap K_2)^* = \mathbf{cl}(K_1^* + K_2^*).$$

*In particular, for any subspace  $L \subseteq \mathbb{R}^n$ ,*

$$(K_1 \cap L)^* = \mathbf{cl}(K_1^* + L^\perp).$$

*Proof.* It is clear that  $K_1^* \cap K_2^* \subseteq (K_1 + K_2)^*$ . To prove the first equality, it is enough to show that  $(K_1 + K_2)^* \subseteq K_1^* \cap K_2^*$ . Let  $l \in (K_1 + K_2)^*$ , then for any  $x^{(1)} \in K_1$ ,  $x^{(2)} \in K_2$ ,  $c_1 > 0$ ,  $c_2 > 0$ , we have  $\langle l, c_1 x^{(1)} + c_2 x^{(2)} \rangle \geq 0$ . Let  $c_1$  and  $c_2$  tend to 0 respectively, we can get  $\langle l, x^{(1)} \rangle \geq 0$  and  $\langle l, x^{(2)} \rangle \geq 0$ , i.e.,  $l \in K_1^* \cap K_2^*$ .

Since  $K_1$  and  $K_2$  are closed, we have  $K_1^{**} = K_1$  and  $K_2^{**} = K_2$ . By the first equality in (2.1), we have

$$(K_1 \cap K_2)^* = (K_1^{**} \cap K_2^{**})^* = ((K_1^* + K_2^*)^*)^* = \mathbf{cl}(K_1^* + K_2^*).$$

□

**Theorem 2.1.2.** [17, Corollary 6.5.1] *Let  $C$  be a convex set and let  $M$  be an affine set which contains a point of  $\mathbf{ri}(C)$ . Then*

$$\mathbf{ri}(M \cap C) = M \cap \mathbf{ri}(C), \quad \mathbf{cl}(M \cap C) = M \cap \mathbf{cl}(C).$$

**Theorem 2.1.3.** [17, Theorem 6.8] *Let  $C$  be a convex set in  $\mathbb{R}^{m+p}$ . For each  $y \in \mathbb{R}^m$ , let  $C_y$  be the set of vectors  $z \in \mathbb{R}^p$  such that  $(y, z) \in C$ . Let  $D = \{y \mid C_y \neq \emptyset\}$ . Then  $(y, z) \in \mathbf{ri}(C)$  if and only if  $y \in \mathbf{ri}(D)$  and  $z \in \mathbf{ri}(C_y)$ .*

**Definition 2.1.4.** A closed convex cone  $K$  is *pointed* if  $K \cap -K = \{0\}$ , i.e.,  $K$  contains no lines through the origin.

**Proposition 2.1.5.** [2, Section 3.3, Exercise 20] *Consider a closed convex cone  $K$  in  $\mathbb{R}^n$ . A base for  $K$  is a convex set with  $0 \notin \mathbf{cl}(C)$  and  $K = \mathbb{R}_+ C$ . The following properties are equivalent:*

- (a)  $K$  is pointed;
- (b)  $\mathbf{cl}(K^* - K^*) = \mathbb{R}^n$ ;
- (c)  $(K^* - K^*) = \mathbb{R}^n$ ;
- (d)  $K^*$  has nonempty interior;
- (e) There exists a vector  $y \in \mathbb{R}^n$  and real  $\epsilon > 0$  with  $\langle y, x \rangle \geq \epsilon \|x\|_2$ , for all points  $x \in K$ ;

(f)  $K$  has a bounded base.

*Proof.* See Appendix.  $\square$

It is well known that the convex hull of a compact set in  $\mathbb{R}^n$  is closed. However, it is generally not true for a non-compact set. For example, let

$$V := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 x_2 = 1, x_2 \geq 0\} \cup \{(0, 0)\},$$

then

$$\mathbf{co}(V) = \{(x_1, x_2) \mid x_1 > 0, x_2 > 0\} \cup \{(0, 0)\},$$

which is not closed.

**Theorem 2.1.6.** *Let  $K$  be a closed cone. The following assertions are equivalent:*

- (a)  $\mathbf{co}(K)$  contains no lines through the origin;
- (b)  $\mathbf{co}(K)$  is closed and pointed;
- (c) There exists a vector  $c = (c_1, \dots, c_n) \in \mathbb{R}^n$  such that  $\langle c, x \rangle > 0$  for all  $x \in \mathbf{co}(K) \setminus \{0\}$ .

*Proof.* By Proposition 2.1.5 (e), it is sufficient to prove (a)  $\Rightarrow$  (b).

Fix a point  $x \in \mathbf{cl}(\mathbf{co}(K))$ , then there is a sequence  $\{x^{(r)}\}_{r=1}^\infty \subseteq \mathbf{co}(K)$  such that  $x^{(r)} \rightarrow x$  as  $r \rightarrow \infty$ . For each  $r \in \mathbb{N}$ , by Carathéodory's Theorem, there exist  $\{x^{(r,l)}\}_{l=1}^{n+1} \subseteq K$  and  $\{\lambda_{r,l}\}_{l=1}^{n+1} \subseteq [0, 1]$  such that  $\sum_{l=1}^{n+1} \lambda_{r,l} = 1$  and

$$(2.2) \quad x^{(r)} = \sum_{l=1}^{n+1} \lambda_{r,l} x^{(r,l)} = \sum_{l=1}^{n+1} \lambda_{r,l} \|x^{(r,l)}\|_2 \frac{x^{(r,l)}}{\|x^{(r,l)}\|_2}.$$

Since the sequence  $\{x^{(r,l)} / \|x^{(r,l)}\|_2\}_{r=1}^\infty$  is bounded for each  $l$ , there exists a subsequence  $x^{(r_t,l)}$  such that

$$(2.3) \quad \lim_{t \rightarrow \infty} \frac{x^{(r_t,l)}}{\|x^{(r_t,l)}\|_2} = y_l, \quad l = 1, \dots, n+1.$$

Because  $K$  is a closed cone, each  $y_l \in K$ . Without loss of generality, we assume (2.3) is true for the whole sequence. In the following, we prove that the sequence  $\{\lambda_{r,l} \|x^{(r,l)}\|_2\}_{r=1}^\infty$  is bounded for each  $l$ .

The closed cone  $\{\sum_{l=1}^n \mu_l y_l \mid \mu_l \geq 0\}$  is pointed since it is contained in  $\mathbf{co}(K)$ . By Proposition 2.1.5 (e), there exists a unit vector  $c \in \mathbb{R}^n$ ,  $\|c\|_2 = 1$  and  $\epsilon > 0$  such that  $\langle c, y_l \rangle > \epsilon$  for each  $1 \leq l \leq n+1$ . Then there exists an  $N \in \mathbb{N}$  such that

$$\frac{\langle c, x^{(r,l)} \rangle}{\|x^{(r,l)}\|_2} > \frac{\epsilon}{2}, \quad \forall r > N, \quad 1 \leq l \leq n+1.$$

Therefore,

$$\begin{aligned}
\|x^{(r)}\|_2 &= \left\| \sum_{l=1}^{n+1} \lambda_{r,l} \|x^{r,l}\|_2 \frac{x^{r,l}}{\|x^{(r,l)}\|_2} \right\|_2 \\
&= \left\| \sum_{l=1}^{n+1} \frac{\lambda_{r,l} \|x^{r,l}\|_2}{\sum_{l=1}^{n+1} \lambda_{r,l} \|x^{r,l}\|_2} \frac{x^{r,l}}{\|x^{(r,l)}\|_2} \left( \sum_{l=1}^{n+1} \lambda_{r,l} \|x^{r,l}\|_2 \right) \right\|_2 \\
&= \|c\|_2 \left\| \sum_{l=1}^{n+1} \frac{\lambda_{r,l} \|x^{r,l}\|_2}{\sum_{l=1}^{n+1} \lambda_{r,l} \|x^{r,l}\|_2} \frac{x^{r,l}}{\|x^{(r,l)}\|_2} \right\|_2 \left( \sum_{l=1}^{n+1} \lambda_{r,l} \|x^{r,l}\|_2 \right) \\
&\geq \left( \sum_{l=1}^{n+1} \frac{\lambda_{r,l} \|x^{r,l}\|_2}{\sum_{l=1}^{n+1} \lambda_{r,l} \|x^{r,l}\|_2} \frac{\langle c, x^{r,l} \rangle}{\|x^{(r,l)}\|_2} \right) \left( \sum_{l=1}^{n+1} \lambda_{r,l} \|x^{r,l}\|_2 \right) \\
&> \frac{\epsilon}{2} \left( \sum_{l=1}^{n+1} \lambda_{r,l} \|x^{r,l}\|_2 \right).
\end{aligned}$$

Since  $x^{(r)} \rightarrow x$  as  $r \rightarrow \infty$ , each sequence  $\{\lambda_{r,l} \|x^{(r,l)}\|_2\}_{r=1}^\infty$  is bounded. There exists a subsequence  $\{\lambda_{r_t,l} \|x^{(r_t,l)}\|_2\}_{t=1}^\infty$  such that

$$\lim_{t \rightarrow \infty} \lambda_{r_t,l} \|x^{(r_t,l)}\|_2 = \mu_l, \quad 1 \leq l \leq n+1$$

for some  $\mu_l$ . Without loss of generality, we assume this is true for the whole sequence. Then

$$\begin{aligned}
x &= \lim_{r \rightarrow \infty} x^{(r)} \\
&= \lim_{r \rightarrow \infty} \sum_{l=1}^{n+1} \lambda_{r,l} \|x^{(r,l)}\|_2 \frac{x^{(r,l)}}{\|x^{(r,l)}\|_2} \\
&= \sum_{l=1}^{n+1} \mu_l y_l \in \mathbf{co}(K).
\end{aligned}$$

Hence  $\mathbf{co}(K)$  is closed and  $\mathbf{cl}(\mathbf{co}(K)) = \mathbf{co}(K)$  contains no line.  $\square$

**Remark 2.1.7.** Although the pointedness is defined on closed convex sets, by Theorem 2.1.6, it is safe to say that  $\mathbf{co}(K)$  is pointed for a closed cone  $K$  if  $\mathbf{co}(K) \cap -\mathbf{co}(K) = \{0\}$ .

**2.2. Quadratic modules and moment matrices.** Let  $\mathbb{N}$  denote the set of non-negative integers and we set  $\mathbb{N}_k^n := \{\alpha \in \mathbb{N}^n \mid |\alpha| = \sum_{i=1}^n \alpha_i \leq k\}$  for  $k \in \mathbb{N}$ . The symbol  $\mathbb{R}[X]$  denotes the ring of multivariate polynomials in variables  $(X_1, \dots, X_n)$  with real coefficients. For  $\alpha \in \mathbb{N}^n$ ,  $X^\alpha$  denotes the monomial  $X_1^{\alpha_1} \cdots X_n^{\alpha_n}$  whose degree is  $|\alpha| := \sum_{i=1}^n \alpha_i$ . The symbol  $\mathbb{R}[X]_k$  denotes the set of real polynomials of degree at most  $k$ .

For any  $p(X) \in \mathbb{R}[X]_k$ , let  $\mathbf{p}$  denote its column vector of coefficients in the monomial basis of  $\mathbb{R}[X]_k$ . A polynomial  $p(X) \in \mathbb{R}[X]$  is said to be a *sum of squares of polynomials* (SOS) if it can be written as  $p(X) = \sum_{i=1}^s u_i(X)^2$  for some  $u_1(X), \dots, u_s(X) \in \mathbb{R}[X]$ . The symbol  $\Sigma^2$  denotes the set of polynomials that are sums of squares.

Let  $G := \{g_1, \dots, g_m\}$  be a set of polynomials that define the semialgebraic set  $S$ . We denote

$$\mathcal{Q}(G) := \left\{ \sum_{j=0}^m \sigma_j g_j \mid g_0 = 1, \sigma_j \in \Sigma^2 \right\}$$

as the *quadratic module* generated by  $G$  and its  $k$ -th *quadratic module*

$$\mathcal{Q}_k(G) := \left\{ \sum_{j=0}^m \sigma_j g_j \mid g_0 = 1, \sigma_j \in \Sigma^2, \deg(\sigma_j g_j) \leq 2k \right\}.$$

It is clear that  $p(x) \geq 0$  for any  $p \in \mathcal{Q}(G)$  and  $x \in S$ .

**Definition 2.2.1.** We say  $\mathcal{Q}(G)$  satisfies the *Archimedean condition* if there exists  $\psi \in \mathcal{Q}(G)$  such that the inequality  $\psi(x) \geq 0$  defines a compact set in  $\mathbb{R}^n$ .

Note that the Archimedean condition implies  $S$  is compact but the inverse is not necessarily true. However, for any compact set  $S$  we can always “force” the associated quadratic module to satisfy the condition by adding a “redundant” constraint  $M - \|x\|_2^2$  for sufficiently large  $M$ .

**Theorem 2.2.2.** [16, PUTINAR’S POSITIVSTELLENSATZ] *Suppose that  $\mathcal{Q}(G)$  satisfies the Archimedean condition. If a polynomial  $p \in \mathbb{R}[X]$  is positive on  $S$ , then  $p \in \mathcal{Q}_k(G)$  for some  $k \in \mathbb{N}$ .*

**Definition 2.2.3.** [10, Definition 3] (Putinar-Prestel’s Bounded Degree Representation of affine polynomials) One says that *Putinar-Prestel’s Bounded Degree Representation* (PP-BDR) of affine polynomials holds for  $S$  if there exists  $k \in \mathbb{N}$  such that if  $p$  is affine and positive on  $S$ , then  $p \in \mathcal{Q}_k(G)$ , except perhaps on a set of vectors  $\mathbf{p} \in \mathbb{R}^n$  with Lebesgue measure zero. Call  $k$  its order.

Let  $y := (y_\alpha)_{\alpha \in \mathbb{N}_{2k}^n}$  be a *truncated moment sequence* of degree  $2k$ . Its associated  $k$ -th *moment matrix* is the matrix  $M_k(y)$  indexed by  $\mathbb{N}_k^n$ , with  $(\alpha, \beta)$ -th entry  $y_{\alpha+\beta}$  for  $\alpha, \beta \in \mathbb{N}_k^n$ . Given a polynomial  $p(X) = \sum_{\alpha} p_{\alpha} X^{\alpha}$ , for  $k \geq d_p = \lceil \deg(p)/2 \rceil$ , the  $(k - d_p)$ -th *localizing moment matrix*  $M_{(k-d_p)}(py)$  is defined as the moment matrix of the *shifted vector*  $((py)_{\alpha})_{\alpha \in \mathbb{N}_{2(k-d_p)}^n}$  with  $(py)_{\alpha} = \sum_{\beta} p_{\beta} y_{\alpha+\beta}$ .  $\mathcal{M}_{2k}$  denotes the space of all truncated moment sequences with degree at most  $2k$ . For any  $y \in \mathcal{M}_{2k}$ , the Riesz functional  $\mathcal{L}_y$  on  $\mathbb{R}[X]_{2k}$  is defined by

$$\mathcal{L}_y \left( \sum_{\alpha} q_{\alpha} X_1^{\alpha_1} \cdots X_n^{\alpha_n} \right) := \sum_{\alpha} q_{\alpha} y_{\alpha}, \quad \forall q(X) \in \mathbb{R}[X]_{2k}.$$

From the definition of the localizing moment matrix  $M_{(k-d_p)}(py)$ , it is easy to check that

$$(2.4) \quad \mathbf{q}^T M_{(k-d_p)}(py) \mathbf{q} = \mathcal{L}_y(p(X)q(X)^2), \quad \forall q(X) \in \mathbb{R}[X]_{k-d_p}.$$

**2.3. Lasserre’s relaxations and theta bodies.** For a compact basic semialgebraic set  $S \subseteq \mathbb{R}^n$ , Lasserre investigated the semidefinite representations of  $\mathbf{co}(S)$  in [10]. Let  $s(k) := \binom{n+k}{n}$  and  $k_j := \lceil \deg g_j / 2 \rceil$  for  $j = 1, \dots, m$ . For any  $k \in \mathbb{N}$ , define

$$(2.5) \quad \Omega_k(G) := \left\{ x \in \mathbb{R}^n \mid \begin{array}{l} \exists y \in \mathbb{R}^{s(2k)}, \text{ s.t. } \mathcal{L}_y(1) = 1, \\ \mathcal{L}_y(X_i) = x_i, \ i = 1, \dots, n, \ M_k(y) \succeq 0, \\ M_{k-k_j}(g_j y) \succeq 0, \ j = 1, \dots, m, \end{array} \right\}.$$

It has been proved in [10, Theorem 2; Theorem 6] that

1. If PP-BDR property holds for  $S$  with order  $k$ , then  $\mathbf{co}(S) = \Omega_k(G)$ ;
2. Assume  $\mathcal{Q}(G)$  is Archimedean. Then for every fixed  $\epsilon > 0$ , there is  $k_\epsilon \in \mathbb{N}$  such that  $\mathbf{co}(S) \subseteq \Omega_{k_\epsilon}(G) \subset \mathbf{co}(S) + \epsilon \mathbf{B}_1$ .

Another hierarchy of semidefinite relaxations of convex hulls closely related to  $\{\Omega_k(G)\}$  is called *theta bodies* defined on real varieties [4, 5], which can be extended to semialgebraic sets. Let  $\mathbb{R}[X]_1$  denote the subset of all linear polynomials in  $\mathbb{R}[X]$ , we have

$$(2.6) \quad \mathbf{cl}(\mathbf{co}(S)) = \bigcap_{p \in \mathbb{R}[X]_1, p|_S \geq 0} \{x \in \mathbb{R}^n \mid p(x) \geq 0\}.$$

Define the  $k$ -th theta body of  $G$  as

$$(2.7) \quad \text{TH}_k(G) := \{x \in \mathbb{R}^n \mid p(x) \geq 0, \quad \forall p \in \mathcal{Q}_k(G) \cap \mathbb{R}[X]_1\}.$$

Clearly, we have

$$\text{TH}_1(G) \supseteq \text{TH}_2(G) \supseteq \cdots \supseteq \text{TH}_k(G) \supseteq \text{TH}_{k+1}(G) \supseteq \cdots \supseteq \mathbf{cl}(\mathbf{co}(S)).$$

When  $\mathcal{Q}(G)$  is Archimedean, by Putinar's Positivstellensatz, (2.6) and (2.7), we have immediately

$$(2.8) \quad \mathbf{cl}(\mathbf{co}(S)) = \bigcap_{k=1}^{\infty} \text{TH}_k(G).$$

**Theorem 2.3.1.** *If  $\mathcal{Q}_k(G)$  is closed, then  $\text{TH}_k(G) = \mathbf{cl}(\Omega_k(G))$ .*

*Proof.* The proof is similar to the one given in [4, Theorem 2.8] for  $S$  being a real variety.  $\square$

The assumption of Archimedean condition plays an essential role in the hierarchy of Lasserre's relaxations (2.5) and theta bodies (2.7). However, for a non-compact semialgebraic set  $S$ , the Archimedean condition is violated. We can not guarantee that the sequence defined in (2.5) or (2.7) converges to  $\mathbf{cl}(\mathbf{co}(S))$ . This can be observed from the following example.

**Example 2.3.2.** Consider the basic semialgebraic set

$$(2.9) \quad S := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0, x_1^2 - x_2^3 \geq 0\}.$$

As shown in Figure 1,  $S$  defines the gray shadow below the right half of the cusp. Let  $G := \{X_1, X_1^2 - X_2^3\}$ . It is clear that the convex hull  $\mathbf{co}(S)$  of  $S$  is itself. We show a tangent line  $l(X_1, X_2) := 1 + 2X_1 - 3X_2 = 0$  of  $S$  at  $(1, 1)$  in Figure 1.

For every  $c_1X_1 + c_2X_2 + c_0 \in \mathcal{Q}_k(G) \cap \mathbb{R}[X]_1$ ,  $c_0, c_1, c_2 \in \mathbb{R}$ , we have

$$c_1X_1 + c_2X_2 + c_0 = \sigma_0(X_1, X_2) + \sigma_1(X_1, X_2)X_1 + \sigma_2(X_1, X_2)(X_1^2 - X_2^3),$$

where  $\sigma_0, \sigma_1, \sigma_2 \in \Sigma^2$ . Substituting  $X_1 = 0$ , we have

$$c_2X_2 + c_0 = \sigma_0(0, X_2) - X_2^3\sigma_2(0, X_2).$$

Since the highest degree terms in  $\sigma_0(0, X_2)$  and  $-X_2^3\sigma_2(0, X_2)$  can not cancel each other out, we have  $\sigma_2(0, X_2) = 0$  and  $\sigma_0(0, X_2)$  is a constant. This implies  $c_2 = 0$  and

$$\text{TH}_k(G) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0\}$$

for all  $k \in \mathbb{N}$ . Hence, theta bodies  $\text{TH}_k(G)$  defined in (2.7) cannot converge to  $\mathbf{co}(S)$ . Moreover, since  $S$  has a nonempty interior,  $\mathcal{Q}_k(G)$  is closed for every  $k \in \mathbb{N}$

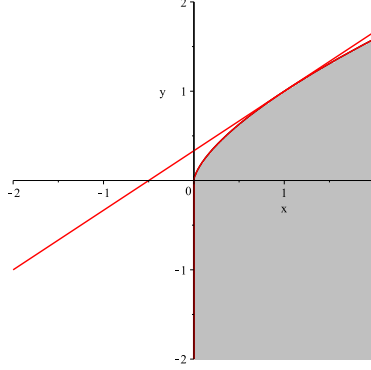


FIGURE 1. The semialgebraic set  $S$  in Example 2.3.2 and the tangent line  $l$ .

[15, 20]. By Theorem 2.3.1, we have  $\text{TH}_k(G) = \text{cl}(\Omega_k(G))$  for  $k \in \mathbb{N}$ . Hence, the hierarchies of Lasserre's relaxations  $\Omega_k(G)$  defined in (2.5) cannot converge to  $\text{co}(S)$ .

The main reason that  $\text{TH}_k(G)$  does not converge to  $\text{co}(S)$  is that none of tangent lines of  $S$ , except  $X_1 = 0$ , can be approximated by polynomials in  $\mathcal{Q}_k(G) \cap \mathbb{R}[X]_1$ ,  $k \in \mathbb{N}$ . In particular,  $l_\epsilon := l + \epsilon \notin \mathcal{Q}_k(G)$  for any  $\epsilon > 0$ ,  $k \in \mathbb{N}$ .  $\square$

**Remark 2.3.3.** Because the semialgebraic sets and projected spectrahedra in all examples in this paper are unbounded, they are shown in figures after being truncated properly.

In next section, we show how to overcome the difficulty in semidefinite representations of convex hulls of non-compact semialgebraic sets.

### 3. SEMIDEFINITE REPRESENTATIONS OF NON-COMPACT CONVEX SETS

In this section, we study how to modify theta bodies and Lasserre's relaxations for computing  $\text{cl}(\text{co}(S))$  when  $S$  is a non-compact semialgebraic set. Our main idea is to lift the cone of  $S$  to a cone of  $\tilde{S}^\circ$  in  $\mathbb{R}^{n+1}$  via homogenization, a technique which has been used in [3, 11] for dealing with non-compact semialgebraic sets, and show that the modified theta bodies and Lasserre's relaxations converge to  $\text{cl}(\text{co}(S))$  when  $S$  is closed at  $\infty$  and  $\text{co}(\text{cl}(\tilde{S}^\circ))$  is closed and pointed. Some examples are given to illustrate that the conditions of pointedness and closedness are essential to ensure the convergence.

**3.1. Nested and closed convex approximations of  $\text{cl}(\text{co}(S))$ .** Consider a polynomial  $f(X) \in \mathbb{R}[X]$  and its homogenization  $\tilde{f}(\tilde{X}) \in \mathbb{R}[\tilde{X}]$ , where  $\tilde{X} = (X_0, X_1, \dots, X_n)$  and  $\tilde{f}(\tilde{X}) = X_0^d f(X/X_0)$ ,  $d = \deg(f)$ . For a given semialgebraic set

$$(3.1) \quad S := \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \dots, g_m(x) \geq 0\},$$



define

$$\begin{aligned}
 \tilde{S}^\circ &:= \{\tilde{x} \in \mathbb{R}^{n+1} \mid \tilde{g}_1(\tilde{x}) \geq 0, \dots, \tilde{g}_m(\tilde{x}) \geq 0, x_0 > 0\}, \\
 \tilde{S}^c &:= \{\tilde{x} \in \mathbb{R}^{n+1} \mid \tilde{g}_1(\tilde{x}) \geq 0, \dots, \tilde{g}_m(\tilde{x}) \geq 0, x_0 \geq 0\}, \\
 \tilde{S} &:= \{\tilde{x} \in \mathbb{R}^{n+1} \mid \tilde{g}_1(\tilde{x}) \geq 0, \dots, \tilde{g}_m(\tilde{x}) \geq 0, x_0 \geq 0, \|\tilde{x}\|_2^2 = 1\}.
 \end{aligned}
 \tag{3.2}$$

**Proposition 3.1.1.** [6, Proposition 2.1]  $f(x) \geq 0$  on  $S$  if and only if  $\tilde{f}(\tilde{x}) \geq 0$  on  $\text{cl}(\tilde{S}^\circ)$ .

**Corollary 3.1.2.** For any  $f \in \mathbb{R}[X]_1$ ,  $f(x) \geq 0$  on  $\text{cl}(\text{co}(S))$  if and only if  $\tilde{f}(\tilde{x}) \geq 0$  on  $\text{co}(\text{cl}(\tilde{S}^\circ))$ .

*Proof.* Since  $f(X)$  is linear,  $f(x) \geq 0$  on  $\text{cl}(\text{co}(S))$  if and only if  $f(x) \geq 0$  on  $S$ , and  $\tilde{f}(\tilde{x}) \geq 0$  on  $\text{co}(\text{cl}(\tilde{S}^\circ))$  if and only if  $\tilde{f}(\tilde{x}) \geq 0$  on  $\text{cl}(\tilde{S}^\circ)$ . The conclusion follows from Proposition 3.1.1.  $\square$

**Definition 3.1.3.** [12]  $S$  is closed at  $\infty$  if  $\text{cl}(\tilde{S}^\circ) = \tilde{S}^c$ .

As pointed out in [6, Remark 2.6], not every semialgebraic set of form (3.1) is closed at  $\infty$ . For instance, it is easy to prove that the set

$$\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2(x_1 - x_2) - 1 = 0, x_1 - 1 \geq 0\}$$

is not closed at  $\infty$ . However, it has been shown in [6] that the closedness at  $\infty$  is a generic property.

Let  $\mathbb{P}[\tilde{X}]_1$  be a set of homogeneous polynomials of degree one in  $\mathbb{R}[\tilde{X}]$  plus the zero polynomial. We define

$$\tilde{G} := \{\tilde{g}_1, \dots, \tilde{g}_m, X_0, \|\tilde{X}\|_2^2 - 1, 1 - \|\tilde{X}\|_2^2\}.$$

We consider the modified theta bodies defined by

$$\widetilde{\text{TH}}_k(\tilde{G}) := \{x \in \mathbb{R}^n \mid \tilde{l}(1, x) \geq 0, \quad \forall \tilde{l} \in \mathcal{Q}_k(\tilde{G}) \cap \mathbb{P}[\tilde{X}]_1\}.$$

Clearly, we have  $\widetilde{\text{TH}}_{k+1}(\tilde{G}) \subseteq \widetilde{\text{TH}}_k(\tilde{G})$  for each  $k \in \mathbb{N}$ .

**Assumption 3.1.4.** (i)  $S$  is closed at  $\infty$ ; (ii) The convex cone  $\text{co}(\text{cl}(\tilde{S}^\circ))$  is closed and pointed.

**Remark 3.1.5.** The condition (ii) is equivalent to the other two conditions in Theorem 2.1.6 and can be verified by them.

**Theorem 3.1.6.** Let  $S \in \mathbb{R}^n$  be the semialgebraic set defined as in (3.1). Suppose that Assumption 3.1.4 is satisfied, then  $\text{cl}(\text{co}(S)) \subseteq \widetilde{\text{TH}}_k(\tilde{G})$  for every  $k \in \mathbb{N}$  and

$$\text{cl}(\text{co}(S)) = \bigcap_{k=1}^{\infty} \widetilde{\text{TH}}_k(\tilde{G}).$$

*Proof.* We first show  $\text{cl}(\text{co}(S)) \subseteq \widetilde{\text{TH}}_k(\tilde{G})$  for every  $k \in \mathbb{N}$ . For an  $\tilde{l} \in \mathcal{Q}_k(\tilde{G}) \cap \mathbb{P}[\tilde{X}]_1$ , we have  $\tilde{l}(\tilde{x}) \geq 0$  on  $\tilde{S}$ . Since  $\tilde{l}$  is homogeneous, we have  $\tilde{l}(\tilde{x}) \geq 0$  on  $\tilde{S}^c$ . Since  $\tilde{S}^\circ \subseteq \tilde{S}^c$ , we have  $\tilde{l}(\tilde{x}) \geq 0$  on  $\text{co}(\text{cl}(\tilde{S}^\circ))$ . By Corollary 3.1.2, we have  $\tilde{l}(1, x) = l(x) \geq 0$  on  $\text{cl}(\text{co}(S))$ , which implies that  $\text{cl}(\text{co}(S))$  is included in

$\widetilde{\text{TH}}_k(\tilde{G})$  for every  $k \in \mathbb{N}$ . Thus, the modified theta bodies defined in (3.4) form a hierarchy of closed convex approximations of  $\mathbf{co}(S)$  as follows:

$$(3.6) \quad \widetilde{\text{TH}}_1(\tilde{G}) \supseteq \widetilde{\text{TH}}_2(\tilde{G}) \supseteq \cdots \supseteq \widetilde{\text{TH}}_k(\tilde{G}) \supseteq \widetilde{\text{TH}}_{k+1}(\tilde{G}) \supseteq \cdots \supseteq \mathbf{cl}(\mathbf{co}(S)).$$

We now verify that this hierarchy converges to  $\mathbf{cl}(\mathbf{co}(S))$  asymptotically. Assume  $u \notin \mathbf{cl}(\mathbf{co}(S))$ , we show that  $u \notin \widetilde{\text{TH}}_k(\tilde{G})$  for some  $k \in \mathbb{N}$ . Since  $\mathbf{cl}(\mathbf{co}(S))$  is closed and convex, by the hyperplane separation theorem, there exists a vector  $(f_0, \mathbf{f}) \in \mathbb{R}^{n+1}$  satisfies

$$\langle \mathbf{f}, u \rangle < f_0 \quad \text{and} \quad \langle \mathbf{f}, x \rangle > f_0 \quad \text{on} \quad \mathbf{cl}(\mathbf{co}(S)).$$

Let  $\tilde{f}(\tilde{X}) := \sum_{i=1}^n f_i X_i - f_0 X_0 \in \mathbb{R}[\tilde{X}]$ , then

$$\tilde{f}(1, u) < 0 \quad \text{and} \quad \tilde{f}(1, x) = f(x) > 0 \quad \text{on} \quad \mathbf{cl}(\mathbf{co}(S)).$$

By Corollary 3.1.2, we have

$$(3.7) \quad \tilde{f}(\tilde{x}) \geq 0 \quad \forall x \in \mathbf{co}(\mathbf{cl}(\tilde{S}^\circ)).$$

Since  $\mathbf{co}(\mathbf{cl}(\tilde{S}^\circ))$  is closed and pointed, by Theorem 2.1.6, there exists a polynomial  $\tilde{g}(\tilde{X}) = \sum_{i=0}^n g_i X_i \in \mathbb{P}[\tilde{X}]_1$  such that  $\tilde{g}(\tilde{x}) > 0$  on  $\mathbf{co}(\mathbf{cl}(\tilde{S}^\circ))$ . We choose a small  $\epsilon > 0$  such that  $(\tilde{f} + \epsilon \tilde{g})(1, u) < 0$  and rename  $\tilde{f} + \epsilon \tilde{g}$  as  $\tilde{f}$ , then

$$(3.8) \quad \tilde{f}(1, u) < 0 \quad \text{and} \quad \tilde{f}(\tilde{x}) > 0 \quad \text{on} \quad \mathbf{cl}(\tilde{S}^\circ).$$

We have assumed that  $S$  is closed at  $\infty$ ,  $\mathbf{cl}(\tilde{S}^\circ) \cap \{\tilde{x} \in \mathbb{R}^{n+1} \mid \|\tilde{x}\|_2 = 1\} = \tilde{S}$ , hence

$$(3.9) \quad \tilde{f}(1, u) < 0 \quad \text{and} \quad \tilde{f}(\tilde{x}) > 0 \quad \text{on} \quad \tilde{S}.$$

Since  $\tilde{S}$  is compact, by Putinar's Positivstellensatz, there exists a  $k' \in \mathbb{N}$  such that  $\tilde{f} \in \mathcal{Q}_{k'}(\tilde{G}) \cap \mathbb{P}[\tilde{X}]_1$ . Since  $\tilde{f}(1, u) < 0$ , we have  $u \notin \widetilde{\text{TH}}_{k'}(\tilde{G})$ . This implies

$$(3.10) \quad \bigcap_{k=1}^{\infty} \widetilde{\text{TH}}_k(\tilde{G}) \subseteq \mathbf{cl}(\mathbf{co}(S)).$$

Finally, by (3.6) and (3.10), we can conclude  $\mathbf{cl}(\mathbf{co}(S)) = \bigcap_{k=1}^{\infty} \widetilde{\text{TH}}_k(\tilde{G})$ .  $\square$

**Example 2.3.2 (continued).** By the definitions (2.9) and (3.2),

$$\tilde{S}^\circ = \{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_1 \geq 0, x_0 x_1^2 - x_2^3 \geq 0, x_0 > 0\},$$

$$\tilde{S}^c = \{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_1 \geq 0, x_0 x_1^2 - x_2^3 \geq 0, x_0 \geq 0\},$$

$$\tilde{S} = \{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_1 \geq 0, x_0 x_1^2 - x_2^3 \geq 0, x_0 \geq 0, \|\tilde{x}\|_2^2 = 1\}.$$

In Figure 2, we show the cone  $\tilde{S}^c$  in  $\mathbb{R}^3$  as well as the hyperplane  $\tilde{l}(X_0, X_1, X_2) := X_0 + 2X_1 - 3X_2 = 0$  generated by  $l$ . It is shown in Figure 3 that  $\tilde{l}$  is nonnegative on  $\tilde{S}$ .

For every  $(0, u_1, u_2) \in \tilde{S}^c \setminus \tilde{S}^\circ$ , let

$$u^{(\epsilon)} := \left( \epsilon, u_1, \sqrt[3]{\epsilon u_1^2 + u_2^3} \right).$$

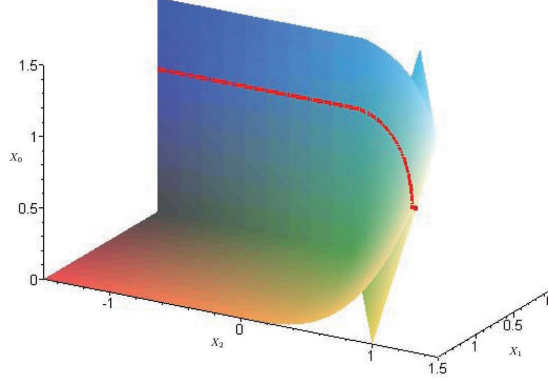


FIGURE 2. The cone  $\tilde{S}^c$  and hyperplane  $\tilde{l}$  generated by  $S$  and  $l$ , respectively, in Example 2.3.2.

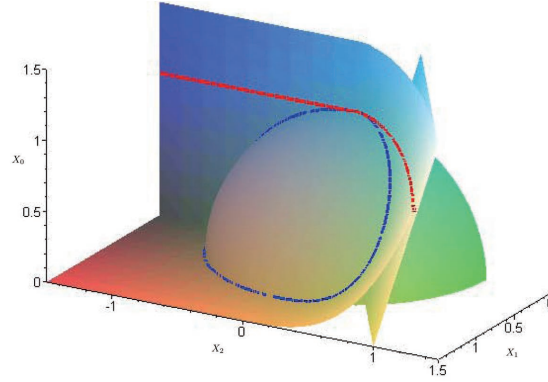


FIGURE 3. The intersection of the cone  $\tilde{S}^c$  and the unit sphere in Example 2.3.2.

Then  $\{u^{(\epsilon)}\}_{\epsilon>0} \subseteq \tilde{S}^\circ$  and  $\lim_{\epsilon \rightarrow 0} u^{(\epsilon)} = (0, u_1, u_2)$ . Hence, we have  $\tilde{S}^c \setminus \tilde{S}^\circ \subseteq \text{cl}(\tilde{S}^\circ)$  and  $S$  is closed at  $\infty$ . Moreover, it can be verified that

$$\tilde{g}(X_0, X_1, X_2) := 2X_0 + 2X_1 - 3X_2$$

is positive on  $\text{co}(\text{cl}(\tilde{S}^\circ)) \setminus \{0\}$  which implies  $\text{co}(\text{cl}(\tilde{S}^\circ))$  is pointed by Theorem 2.1.6. Hence Assumption 3.1.4 holds for  $S$ .

Let  $\epsilon > 0$  tend to 0,  $\tilde{l}$  can be approximated by  $\tilde{l} + \epsilon\tilde{g}$  which are positive on  $\tilde{S}$ . Moreover, since  $\tilde{S}$  is compact, by Putinar's Positivstellensatz,  $\tilde{l} + \epsilon\tilde{g}$  belongs to the quadratic module corresponding to

$$\tilde{G} := \{X_0, X_1, X_0X_1^2 - X_2^3, X_0^2 + X_1^2 + X_2^2 - 1, 1 - X_0^2 - X_1^2 - X_2^2\}$$

for every  $\epsilon > 0$ . Define

$$\widetilde{\text{TH}}_k(\tilde{G}) := \{(x_1, x_2) \in \mathbb{R}^2 \mid \tilde{l}(1, x_1, x_2) \geq 0, \quad \forall \tilde{l} \in \mathcal{Q}_k(\tilde{G}) \cap \mathbb{P}[X_0, X_1, X_2]_1\},$$

we have

$$\mathbf{cl}(\mathbf{co}(S)) = \bigcap_{k=1}^{\infty} \widetilde{\mathbf{TH}}_k(\tilde{G}).$$

**Corollary 3.1.7.** *Let  $S \in \mathbb{R}^n$  be a semialgebraic set defined as in (3.1). Suppose that Assumption 3.1.4 is satisfied and PP-BDR property holds for  $\tilde{S}$  with order  $k'$ , then  $\mathbf{cl}(\mathbf{co}(S)) = \widetilde{\mathbf{TH}}_{k'}(\tilde{G})$ .*

*Proof.* Suppose that  $\tilde{S}$  satisfies PP-BDR property with order  $k'$ , for every  $\tilde{f}(\tilde{x}) > 0$  on  $\tilde{S}$ , we have  $\tilde{f} \in \mathcal{Q}_{k'}(\tilde{G})$ . The inclusion  $\widetilde{\mathbf{TH}}_{k'}(\tilde{G}) \supseteq \mathbf{cl}(\mathbf{co}(S))$  is obvious by (3.6). Now we verify  $\mathbf{cl}(\mathbf{co}(S)) \supseteq \widetilde{\mathbf{TH}}_{k'}(\tilde{G})$ .

Assume that there exists a vector  $u \in \widetilde{\mathbf{TH}}_{k'}(\tilde{G})$  but  $u \notin \mathbf{cl}(\mathbf{co}(S))$ . According to (3.9), there exists a linear polynomial  $\tilde{f} \in \mathbb{R}[\tilde{X}]$  with  $\tilde{f}(0) = 0$  such that  $\tilde{f}(1, u) < 0$  and  $\tilde{f}(\tilde{x}) > 0$  on  $\tilde{S}$ . Since  $\tilde{S}$  satisfies PP-BDR property with order  $k'$ , we have  $\tilde{f} \in \mathcal{Q}_{k'}(\tilde{G})$ . Due to the fact that  $\tilde{f}(1, u) < 0$ , we derive that  $u \notin \widetilde{\mathbf{TH}}_{k'}(\tilde{G})$ . This yields the contradiction. Thus, we have  $\mathbf{cl}(\mathbf{co}(S)) = \widetilde{\mathbf{TH}}_{k'}(\tilde{G})$ .  $\square$

We would like to point out that two conditions in Assumption 3.1.4 can not be dropped in Theorem 3.1.6.

**Example 3.1.8.** Consider the semialgebraic set  $S := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 - x_1^2 \geq 0\}$ . Clearly,  $\mathbf{cl}(\mathbf{co}(S)) = S$ . We have

$$\begin{aligned} \tilde{S}^o &= \{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_0 x_2 - x_1^2 \geq 0, x_0 > 0\}, \\ \tilde{S}^c &= \{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_0 x_2 - x_1^2 \geq 0, x_0 \geq 0\}. \end{aligned}$$

It is easy to check that  $\mathbf{cl}(\tilde{S}^o)$  is convex. Define  $\tilde{f}(\tilde{X}) := X_0 + X_2$ , we have  $\tilde{f}(\tilde{x}) > 0$  on  $\mathbf{co}(\mathbf{cl}(\tilde{S}^o)) \setminus \{0\}$  which implies  $\mathbf{co}(\mathbf{cl}(\tilde{S}^o))$  is closed and pointed. However,  $\tilde{S}^c \setminus \mathbf{cl}(\tilde{S}^o) = \{(0, 0, x_2) \in \mathbb{R}^3 \mid x_2 < 0\} \neq \emptyset$  means  $S$  is not closed at  $\infty$ . Let

$$\tilde{G} = \{X_0, X_0 X_2 - X_1^2, X_0^2 + X_1^2 + X_2^2 - 1, 1 - X_0^2 - X_1^2 - X_2^2\}.$$

We prove that  $\widetilde{\mathbf{TH}}_k(\tilde{G}) = \mathbb{R}^2$  for every  $k \in \mathbb{N}$  and  $\mathbf{cl}(\mathbf{co}(S)) \neq \bigcap_{k=1}^{\infty} \widetilde{\mathbf{TH}}_k(\tilde{G}) = \mathbb{R}^2$ .

Assume  $c_0 X_0 + c_1 X_1 + c_2 X_2 \in \mathcal{Q}_k(\tilde{G})$ , then

$$(3.11) \quad c_0 X_0 + c_1 X_1 + c_2 X_2 = \tilde{\sigma}_0 + \tilde{\sigma}_1 X_0 + \tilde{\sigma}_2 (X_0 X_2 - X_1^2) + \tilde{h} (X_0^2 + X_1^2 + X_2^2 - 1),$$

where  $\tilde{\sigma}_i \in \Sigma^2$ ,  $i = 0, 1, 2$  and  $\tilde{h} \in \mathbb{R}[\tilde{X}]$ . By substituting  $(X_0, X_1, X_2) = (0, 0, \pm 1)$  in (3.11), we get  $\pm c_2 = \tilde{\sigma}_0(0, 0, \pm 1) \geq 0$  which implies  $c_2 = 0$ . Assume  $c_1 \neq 0$  and let

$$x^{(1)} = \left(1, \frac{-c_0 + c_1}{c_1}, \frac{(-c_0 + c_1)^2}{c_1^2}\right), \quad x^{(2)} = \left(1, -\frac{c_0 + c_1}{c_1}, \frac{(c_0 + c_1)^2}{c_1^2}\right).$$

Let  $c_2 = 0$  and substitute  $(X_0, X_1, X_2)$  by  $x^{(1)}/\|x^{(1)}\|_2$  and  $x^{(2)}/\|x^{(2)}\|_2$  in (3.11) respectively, we get

$$\begin{aligned} c_1 &= \|x^{(1)}\|_2 \left( \tilde{\sigma}_0 \left( \frac{x^{(1)}}{\|x^{(1)}\|_2} \right) + \tilde{\sigma}_1 \left( \frac{x^{(1)}}{\|x^{(1)}\|_2} \right) \frac{1}{\|x^{(1)}\|_2} \right) \geq 0, \\ -c_1 &= \|x^{(2)}\|_2 \left( \tilde{\sigma}_0 \left( \frac{x^{(2)}}{\|x^{(2)}\|_2} \right) + \tilde{\sigma}_1 \left( \frac{x^{(2)}}{\|x^{(2)}\|_2} \right) \frac{1}{\|x^{(2)}\|_2} \right) \geq 0, \end{aligned}$$

which implies  $c_1 = 0$ . It contradicts the assumption  $c_1 \neq 0$ . Hence, we have  $c_1 = 0$ . By the definition in (3.4), we get  $\widetilde{\text{TH}}_k(\widetilde{G}) = \mathbb{R}^2$  for every  $k \in \mathbb{N}$ . Therefore, we conclude that the assumption of closedness of  $S$  at  $\infty$  can not be dropped in Theorem 3.1.6.  $\square$

**Example 3.1.9.** Consider the set  $S := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2^3 - x_1^2 \geq 0\}$ . We have  $\text{cl}(\text{co}(S)) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq 0\}$  and

$$\begin{aligned}\widetilde{S}^o &= \{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_2^3 - x_0x_1^2 \geq 0, x_0 > 0\}, \\ \widetilde{S}^c &= \{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_2^3 - x_0x_1^2 \geq 0, x_0 \geq 0\}.\end{aligned}$$

It can be verified that  $\widetilde{S}^c \setminus \widetilde{S}^o = \{(0, x_1, x_2) \in \mathbb{R}^3 \mid x_2 \geq 0\}$ . Using similar arguments in **Example 2.3.2 (continued)**, we can show that  $S$  is closed at  $\infty$ . However,  $\lim_{\epsilon \rightarrow 0}(\epsilon, \pm 1, \sqrt[3]{\epsilon}) = (0, \pm 1, 0)$  and  $(0, \pm 1, 0) \in \text{cl}(\widetilde{S}^o)$ , which implies that  $\text{co}(\text{cl}(\widetilde{S}^o))$  is not pointed.

Let

$$\widetilde{G} = \{X_0, X_2^3 - X_0X_1^2, X_0^2 + X_1^2 + X_2^2 - 1, 1 - X_0^2 - X_1^2 - X_2^2\},$$

we show  $\widetilde{\text{TH}}_k(\widetilde{G}) = \mathbb{R}^2$  for every  $k \in \mathbb{N}$ .

Assume  $c_0X_0 + c_1X_1 + c_2X_2 \in \mathcal{Q}_k(\widetilde{G})$ , then

$$(3.12) \quad c_0X_0 + c_1X_1 + c_2X_2 = \tilde{\sigma}_0 + \tilde{\sigma}_1X_0 + \tilde{\sigma}_2(X_2^3 - X_0X_1^2) + \tilde{h}(X_0^2 + X_1^2 + X_2^2 - 1),$$

$\tilde{\sigma}_i \in \Sigma^2, i = 0, 1, 2$  and  $\tilde{h} \in \mathbb{R}[\widetilde{X}]$ . Substituting  $(X_0, X_1, X_2) = (0, \pm 1, 0)$  in (3.12), we derive  $c_1 = 0$ . Substituting  $(X_0, X_1, X_2) = (0, \pm 1, X_2)$  in (3.12), we have

$$(3.13) \quad c_2X_2 = \tilde{\sigma}_0(0, \pm 1, X_2) + \tilde{\sigma}_2(0, \pm 1, X_2)X_2^3 + \tilde{h}(0, \pm 1, X_2)X_2^2.$$

It is clear that  $\tilde{\sigma}_0(0, \pm 1, X_2)$  can not have a nonzero constant term. Hence, the right side of the equation (3.13) is divisible by  $X_2^2$ , which is only possible when  $c_2 = 0$ . By the definition of the theta body, we derive  $\widetilde{\text{TH}}_k(\widetilde{G}) = \mathbb{R}^2$  for every  $k \in \mathbb{N}$ . This shows that the assumption of pointedness of  $\text{co}(\text{cl}(\widetilde{S}^o))$  can not be dropped in Theorem 3.1.6.  $\square$

Since the PP-BDR property is not generally true, similar to Lemma 5 and Theorem 6 in [10, Section 2.5], we give an approximate semidefinite representation of  $\text{cl}(\text{co}(S))$ . For a radius  $r \in \mathbb{R}$ , let  $\mathbf{B}_r := \{x \in \mathbb{R}^n \mid \|x\|_2 \leq r\}$ .

**Lemma 3.1.10.** *Let  $\Omega \subset \mathbb{R}^n$  be a closed convex set and let  $r > 0, \epsilon > 0$  be fixed. Assume that  $(\Omega + \epsilon\mathbf{B}_1) \cap \mathbf{B}_r \neq \emptyset$  and  $u \in \mathbf{B}_r \setminus (\Omega + \epsilon\mathbf{B}_1)$ , then there exists a unit vector  $\mathbf{f} \in \mathbb{R}^n$  and a scalar  $f^*$  with  $|f^*| \leq 3r + \epsilon$  such that*

$$(3.14) \quad \mathbf{f}^T x \geq f^* \quad \forall x \in \Omega \quad \text{and} \quad \mathbf{f}^T u < f^* - \epsilon.$$

*Proof.* Since  $\Omega$  is closed and convex, there is a unique projection  $u^*$  of  $u$  on  $\Omega$ . Let  $\mathbf{f} := (u^* - u)/\|u - u^*\|_2$  and  $f^* := \mathbf{f}^T u^*$ . Using the same arguments in the proof of [10, Lemma 5], we conclude that (3.14). Moreover, let  $\bar{u} \in (\Omega + \epsilon\mathbf{B}_1) \cap \mathbf{B}_r$ , there

exists  $\hat{u} \in \Omega$  such that  $\|\bar{u} - \hat{u}\|_2 \leq \epsilon$ . Hence, we have

$$\begin{aligned}
|f^*| &\leq \|\mathbf{f}\|_2 \|u^*\|_2 \\
&\leq \|u\|_2 + \|u^* - u\|_2 \\
&\leq \|u\|_2 + \|\hat{u} - u\|_2 \\
&\leq \|u\|_2 + \|\hat{u} - \bar{u}\|_2 + \|\bar{u} - u\|_2 \\
&\leq r + \epsilon + 2r = 3r + \epsilon
\end{aligned}$$

□

**Theorem 3.1.11.** *Let  $S \in \mathbb{R}^n$  be a semialgebraic set defined as in (3.1). Suppose that Assumption 3.1.4 holds, then for every fixed  $\epsilon > 0$  and  $r > 0$  with  $\mathbf{cl}(\mathbf{co}(S)) \cap \mathbf{B}_r \neq \emptyset$ , there exists an integer  $k_{r,\epsilon} \in \mathbb{N}$  such that*

$$\mathbf{cl}(\mathbf{co}(S)) \cap \mathbf{B}_r \subseteq \widetilde{\text{TH}}_{k_{r,\epsilon}}(\tilde{G}) \cap \mathbf{B}_r \subseteq (\mathbf{cl}(\mathbf{co}(S)) + \epsilon \mathbf{B}_1) \cap \mathbf{B}_r$$

holds.

*Proof.* By Theorem 3.1.6, we only need to prove

$$(3.15) \quad \widetilde{\text{TH}}_{k_{r,\epsilon}}(\tilde{G}) \cap \mathbf{B}_r \subseteq (\mathbf{cl}(\mathbf{co}(S)) + \epsilon \mathbf{B}_1) \cap \mathbf{B}_r.$$

Without loss of generality, we assume  $(\mathbf{cl}(\mathbf{co}(S)) + \epsilon \mathbf{B}_1) \cap \mathbf{B}_r \neq \mathbf{B}_r$  and let  $u \in \mathbf{B}_r \setminus (\mathbf{cl}(\mathbf{co}(S)) + \epsilon \mathbf{B}_1)$  be fixed. By Lemma 3.1.10, there exists  $\mathbf{f} \in \mathbb{R}^n$  with  $\|\mathbf{f}\|_2 = 1$  and a scalar  $f^*$  with  $|f^*| \leq 3r + \epsilon$  such that the following condition holds

$$\mathbf{f}^T x \geq f^* \quad \text{on} \quad \mathbf{cl}(\mathbf{co}(S)) \quad \text{and} \quad \mathbf{f}^T u < f^* - \epsilon.$$

Define  $\tilde{f}(\tilde{X}) := \sum_{i=0}^n f_i X_i - f^* X_0 \in \mathbb{R}[\tilde{X}]$ , by Corollary 3.1.2,  $\tilde{f}(\tilde{x}) \geq 0$  on  $\mathbf{co}(\mathbf{cl}(\tilde{S}^\circ))$ . Since  $\mathbf{co}(\mathbf{cl}(\tilde{S}^\circ))$  is closed and pointed, by Theorem 2.1.6, there exists a polynomial  $\tilde{g}(\tilde{X}) = \sum_{i=0}^n g_i X_i \in \mathbb{P}[\tilde{X}]_1$  such that  $\|\tilde{\mathbf{g}}\|_2 = 1$  and  $\tilde{g}(\tilde{x}) > 0$  on  $\mathbf{co}(\mathbf{cl}(\tilde{S}^\circ))$ . We define a new polynomial

$$\tilde{p}(\tilde{X}) := \frac{\epsilon}{\sqrt{1+r^2}} \tilde{g}(\tilde{X}) + \tilde{f}(\tilde{X}) \in \mathbb{R}[\tilde{X}],$$

then

$$\begin{aligned}
\|\tilde{\mathbf{p}}\|_2 &\leq \frac{\epsilon}{\sqrt{1+r^2}} \|\tilde{\mathbf{g}}\|_2 + \|\tilde{\mathbf{f}}\|_2 \\
&\leq \frac{\epsilon}{\sqrt{1+r^2}} + 1 + 3r + \epsilon,
\end{aligned}$$

and

$$\begin{aligned}
\tilde{p}(1, u) &= \frac{\epsilon}{\sqrt{1+r^2}} \tilde{g}(1, u) + \tilde{f}(1, u) \\
&\leq \frac{\epsilon}{\sqrt{1+r^2}} \|\tilde{\mathbf{g}}\|_2 \|(1, u)\|_2 + \mathbf{f}^T u - f^* \\
&< \epsilon - \epsilon = 0.
\end{aligned}$$

Let

$$c := \min \left\{ \tilde{g}(\tilde{x}) \mid \tilde{x} \in \mathbf{cl}(\tilde{S}^\circ), \|\tilde{x}\|_2 = 1 \right\},$$

then  $c > 0$  is well defined and  $\tilde{p}(\tilde{x}) \geq c\epsilon/\sqrt{1+r^2} > 0$  on  $\mathbf{cl}(\tilde{S}^\circ) \cap \{\|\tilde{x}\|_2 = 1\}$ .

As  $S$  is closed at  $\infty$ , we have  $\tilde{p}(\tilde{x}) \geq c\epsilon/\sqrt{1+r^2} > 0$  on  $\tilde{S}$ . Since  $\tilde{S}$  is compact, by Putinar's Positivstellensatz [16] and [13, Theorem 6], there exists an integer

$k_{r,\epsilon} \in \mathbb{N}$  depending only on  $r$  and  $\epsilon$  such that  $\tilde{p} \in \mathcal{Q}_{k_{r,\epsilon}}(\tilde{G})$ . From  $\tilde{p}(1, u) < 0$ , we derive  $u \notin \widetilde{\text{TH}}_{k_{r,\epsilon}}(\tilde{G})$ . This implies (3.15).  $\square$

**3.2. Spectrahedral approximations of  $\text{cl}(\text{co}(S))$ .** In order to fulfill computations of  $\text{cl}(\text{co}(S))$  via semidefinite programming, we study an alternative description of  $\widetilde{\text{TH}}_k(\tilde{G})$  in a dual view and establish the connection between them. In the following, we consider moment sequences  $y$  of real numbers indexed by  $(n+1)$ -tuple  $\alpha := (\alpha_0, \alpha_1, \dots, \alpha_n) \in \mathbb{N}^{n+1}$ . Each  $y$  defines a Riesz functional  $\mathcal{L}_y$  on  $\mathbb{R}[\tilde{X}]$ . Recall that  $\tilde{s}(k) = \binom{n+k+1}{n+1}$  and  $k_j = \lceil \deg g_j / 2 \rceil$ . For every  $k \in \mathbb{N}$ , define

$$(3.16) \quad \Omega_k(\tilde{G}) := \left\{ x \in \mathbb{R}^n \left| \begin{array}{l} \exists y \in \mathbb{R}^{\tilde{s}(2k)}, \text{ s.t. } \mathcal{L}_y(X_0) = 1, \\ \mathcal{L}_y(X_i) = x_i, \quad i = 1, \dots, n, \\ M_{k-1}(X_0 y) \succeq 0, \quad M_{k-1}((\|\tilde{X}\|_2^2 - 1)y) = 0, \\ M_k(y) \succeq 0, \quad M_{k-k_j}(\tilde{g}_j y) \succeq 0, \quad j = 1, \dots, m \end{array} \right. \right\}.$$

**Theorem 3.2.1.** *We have  $\text{cl}(\text{co}(S)) \subseteq \text{cl}(\Omega_k(\tilde{G})) \subseteq \widetilde{\text{TH}}_k(\tilde{G})$  for every  $k \in \mathbb{N}$ .*

*Proof.* Since  $\widetilde{\text{TH}}_k(\tilde{G})$  is closed, it is sufficient to prove

$$\text{co}(S) \subseteq \Omega_k(\tilde{G}) \subseteq \widetilde{\text{TH}}_k(\tilde{G}) \quad \text{for each } k \in \mathbb{N}.$$

Fixing a vector  $u \in S$ , let  $\tilde{u} := (1, u) / \|(1, u)\|_2 \in \tilde{S}$  and  $y := \{y_\alpha\}_{|\alpha| \leq 2k}$ , where  $y_\alpha = \tilde{u}^\alpha / \tilde{u}_0$  for  $\alpha \in \mathbb{N}^{n+1}$ . It is quite straightforward to show that  $\mathcal{L}_y(X_0) = 1$ ,  $\mathcal{L}_y(X_i) = u_i$ ,  $i = 1, \dots, n$  and

$$\begin{aligned} \langle \mathbf{w}, M_k(y) \mathbf{w} \rangle &= \frac{1}{\tilde{u}_0} w^2(\tilde{u}) \geq 0, \quad \forall w \in \mathbb{R}[\tilde{X}]_k, \\ \langle \mathbf{v}, M_{k-1}(X_0 y) \mathbf{v} \rangle &= \frac{1}{\tilde{u}_0} \tilde{u}_0 v^2(\tilde{u}) = v^2(\tilde{u}) \geq 0, \quad \forall v \in \mathbb{R}[\tilde{X}]_{k-1}, \\ \langle \mathbf{p}, M_{k-1}((\|\tilde{X}\|_2^2 - 1)y) \mathbf{p} \rangle &= \frac{1}{\tilde{u}_0} p^2(\tilde{u}) (\|\tilde{u}\|_2^2 - 1) = 0, \quad \forall p \in \mathbb{R}[\tilde{X}]_{k-1}, \\ \langle \mathbf{q}, M_{k-k_j}(\tilde{g}_j y) \mathbf{q} \rangle &= \frac{1}{\tilde{u}_0} q^2(\tilde{u}) \tilde{g}_j(\tilde{u}) \geq 0, \quad \forall q \in \mathbb{R}[\tilde{X}]_{k-k_j}, \quad j = 1, \dots, m. \end{aligned}$$

Therefore, we derive  $u \in \Omega_k(\tilde{G})$  and  $S \subseteq \Omega_k(\tilde{G})$ . Since  $\Omega_k(\tilde{G})$  is convex, it is clear that we have  $\text{co}(S) \subseteq \Omega_k(\tilde{G})$ .

For a given  $v \in \Omega_k(\tilde{G})$ , let  $y \in \mathbb{R}^{\tilde{s}(2k)}$  be its associated moment sequence defined in (3.16). For every  $\tilde{l} \in \mathcal{Q}_k(\tilde{G}) \cap \mathbb{P}[\tilde{X}]_1$ , we have the representation

$$\tilde{l}(\tilde{X}) = \tilde{\sigma} + \tilde{\sigma}_0 X_0 + \tilde{h}(\|\tilde{X}\|_2^2 - 1) + \sum_{j=1}^m \tilde{\sigma}_j \tilde{g}_j,$$

where  $\tilde{\sigma}_j$ 's are SOS and each term in the summation has degree  $\leq 2k$ . We have

$$\tilde{l}(1, v) = \mathcal{L}_y(\tilde{l}) = \mathcal{L}_y \left( \tilde{\sigma} + \tilde{\sigma}_0 X_0 + \tilde{h}(\|\tilde{X}\|_2^2 - 1) + \sum_{j=1}^m \tilde{\sigma}_j \tilde{g}_j \right)$$

By (2.4) and (3.16), we obtain  $\tilde{l}(1, v) \geq 0$ , which implies  $v \in \widetilde{\text{TH}}_k(\tilde{G})$ . The proof is completed.  $\square$

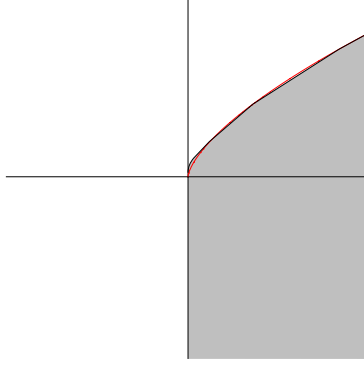


FIGURE 4. The spectrahedral approximation  $\Omega_3(\tilde{G})$  (shown shaded) of  $\mathbf{cl}(\mathbf{co}(S))$  in Example 2.3.2.

**Example 2.3.2 (continued).** Using the software package Bermeja [18], we draw the third order spectrahedron  $\Omega_3(\tilde{G})$ . As shown in Figure 4, our modified Lasserre's relaxation  $\Omega_3(\tilde{G})$  is a very tight approximation of  $\mathbf{cl}(\mathbf{co}(S))$ .  $\square$

The following results are obtained by replacing  $\widetilde{\text{TH}}_k(\tilde{G})$  by  $\mathbf{cl}(\Omega_k(\tilde{G}))$  in Theorem 3.1.6, Corollary 3.1.7 and Theorem 3.1.11.

**Corollary 3.2.2.** *Let  $S \in \mathbb{R}^n$  be a semialgebraic set defined as in (3.1). Suppose that Assumption 3.1.4 is satisfied, then*

1.  $\mathbf{cl}(\mathbf{co}(S)) \subseteq \Omega_k(\tilde{G})$  for every  $k \in \mathbb{N}$  and  $\mathbf{cl}(\mathbf{co}(S)) = \bigcap_{k=1}^{\infty} \mathbf{cl}(\Omega_k(\tilde{G}))$ .
2. If PP-BDR property holds for  $\tilde{S}$  with order  $k'$ , then  $\mathbf{cl}(\mathbf{co}(S)) = \mathbf{cl}(\Omega_{k'}(\tilde{G}))$ .
3. For every fixed  $\epsilon > 0$  and  $r > 0$  with  $\mathbf{cl}(\mathbf{co}(S)) \cap \mathbf{B}_r \neq \emptyset$ , there exists an integer  $k_{r,\epsilon} \in \mathbb{N}$  such that

$$\mathbf{cl}(\mathbf{co}(S)) \cap \mathbf{B}_r \subseteq \mathbf{cl}(\Omega_{k_{r,\epsilon}}(\tilde{G})) \cap \mathbf{B}_r \subseteq (\mathbf{cl}(\mathbf{co}(S)) + \epsilon \mathbf{B}_1) \cap \mathbf{B}_r$$

holds.

*Proof.* It is straightforward to verify these results via the preceding theorem.  $\square$

**Example 3.2.3.** Consider the following semialgebraic set

$$S := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^3 - x_2^2 - x_1 + 1 = 0, x_2 \geq 0\},$$

which is the red curve shown in Figure 5. We have

$$\tilde{S}^o = \{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_1^3 - x_0 x_2^2 - x_0^2 x_1 + x_0^3 = 0, x_2 \geq 0, x_0 > 0\},$$

$$\tilde{S}^c = \{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_1^3 - x_0 x_2^2 - x_0^2 x_1 + x_0^3 = 0, x_2 \geq 0, x_0 \geq 0\}.$$

Clearly,  $\tilde{S}^c \setminus \tilde{S}^o = \{(0, 0, x_2) \in \mathbb{R}^3 \mid x_2 \geq 0\}$ . It can be verified that  $\tilde{S}^c = \mathbf{cl}(\tilde{S}^o)$ ,

i.e.,  $S$  is closed at  $\infty$ . Since  $X_0 + X_2 > 0$  on  $\mathbf{cl}(\tilde{S}^o) \setminus \{0\}$ , we have  $X_0 + X_2 > 0$  on  $\mathbf{co}(\mathbf{cl}(\tilde{S}^o)) \setminus \{0\}$  and thus, by Theorem 2.1.6,  $\mathbf{co}(\mathbf{cl}(\tilde{S}^o))$  is closed and pointed.

Hence, Assumption 3.1.4 holds for  $S$ . The third projected spectrahedron  $\Omega_3(\tilde{G})$  is depicted (shaded) in Figure 5.  $\square$



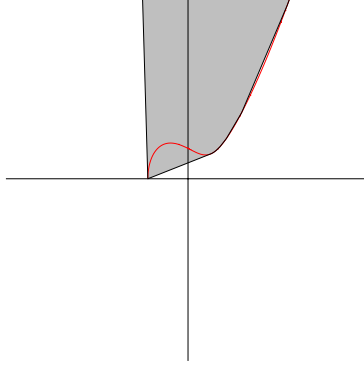


FIGURE 5. The semialgebraic set  $S$  (red curve) and projected spectrahedron  $\Omega_3(\tilde{G})$  (shaded) in Example 3.2.3.

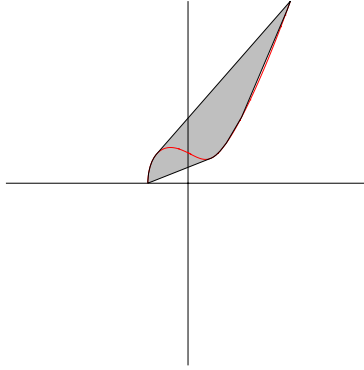


FIGURE 6. Lasserre's relaxation of  $\mathbf{cl}(\mathbf{co}(S)) \cap \mathbf{B}_r$  in Example 3.2.3.

**Remark 3.2.4.** As shown in Theorem 3.1.11 and Corollary 3.2.2, the projected spectrahedra  $\Omega_k(\tilde{G})$  for  $k \in \mathbb{N}$  are outer approximations of  $\mathbf{cl}(\mathbf{co}(S))$  and convergent uniformly to  $\mathbf{cl}(\mathbf{co}(S))$  restricted to every fixed ball  $\mathbf{B}_r$ . If we truncated  $S$  first by the ball  $\mathbf{B}_r$  and then compute the convex hull of the resulting compact set by Lasserre's relaxations (2.5), in general, we can not get approximations of the truncation  $\mathbf{cl}(\mathbf{co}(S)) \cap \mathbf{B}_r$ . Taking Example 3.2.3 for instance, compared with Figure 5, Lasserre's relaxation of  $\mathbf{cl}(\mathbf{co}(S \cap \mathbf{B}_r))$  shown in Figure 6 is not an outer approximation of the truncation  $\mathbf{cl}(\mathbf{co}(S)) \cap \mathbf{B}_r$ .

By similar arguments given in [4, 5], we show below that  $\widetilde{\mathbf{TH}}_k(\tilde{G}) = \mathbf{cl}(\Omega_k(\tilde{G}))$  for each  $k \in \mathbb{N}$  if  $\mathcal{Q}_k(\tilde{G})$  is closed.

Let  $M := \{(1, x) \mid x \in \mathbb{R}^n\}$  and  $\mathcal{Q}_k^1(\tilde{G}) = \mathcal{Q}_k(\tilde{G}) \cap \mathbb{P}[\tilde{X}]_1$ . By the definition of dual cones,

$$(3.17) \quad \mathcal{Q}_k^1(\tilde{G})^* \cap M = \{1\} \times \widetilde{\mathbf{TH}}_k(\tilde{G}).$$

Denote  $\mathbf{proj}(\mathcal{Q}_k(\tilde{G})^*)$  as the projection of  $\mathcal{Q}_k(\tilde{G})^*$  onto  $(\mathbb{P}[\tilde{X}]_1)^*$ . It is clear that

$$(3.18) \quad \mathbf{proj}(\mathcal{Q}_k(\tilde{G})^*) \cap M = \{1\} \times \Omega_k(\tilde{G}).$$

If  $\mathcal{Q}_k(\tilde{G})$  is closed, by Proposition 2.1.1, we have

$$(3.19) \quad \mathcal{Q}_k^1(\tilde{G})^* = \mathbf{cl}(\mathbf{proj}(\mathcal{Q}_k(\tilde{G})^*)).$$

**Lemma 3.2.5.** *If  $\mathcal{Q}_k(\tilde{G})$  is closed, the hyperplane  $M$  intersects  $\mathbf{ri}(\mathbf{proj}(\mathcal{Q}_k(\tilde{G})^*))$ .*

*Proof.* By [17, Theorem 6.3] and (3.19), it is equivalent to prove  $M$  intersects  $\mathbf{ri}(\mathcal{Q}_k^1(\tilde{G})^*)$ . Fixing a vector  $u \in S$ , then we have  $\tilde{l} := (1, u)/\|(1, u)\|_2 \in \tilde{S}$  and  $\tilde{l} \in \mathcal{Q}_k^1(\tilde{G})^*$ . Let

$$D := \{t_0 \in \mathbb{R} \mid \exists t \in \mathbb{R}^n, \text{ s.t. } (t_0, t) \in \mathcal{Q}_k^1(\tilde{G})^*\}.$$

Since  $X_0 \in \mathcal{Q}_k^1(\tilde{G})$  and  $c \cdot \tilde{l} \in \mathcal{Q}_k^1(\tilde{G})^*$  for all  $c \geq 0$ , we get  $D = [0, \infty)$  and thus  $1 \in \mathbf{ri}(D)$ . By Theorem 2.1.3, we derive that  $M$  intersects  $\mathbf{ri}(\mathcal{Q}_k^1(\tilde{G})^*)$ .  $\square$

**Theorem 3.2.6.** *If  $\mathcal{Q}_k(\tilde{G})$  is closed, then  $\widetilde{\text{TH}}_k(\tilde{G}) = \mathbf{cl}(\Omega_k(\tilde{G}))$ .*

*Proof.* By (3.17), (3.18), (3.19), Theorem 2.1.2 and Lemma 3.2.5, we have

$$\begin{aligned} \{1\} \times \mathbf{cl}(\Omega_k(\tilde{G})) &= \mathbf{cl}(\mathbf{proj}(\mathcal{Q}_k(\tilde{G})^*) \cap M) \\ &= \mathbf{cl}(\mathbf{proj}(\mathcal{Q}_k(\tilde{G})^*)) \cap M \\ &= \mathcal{Q}_k^1(\tilde{G})^* \cap M \\ &= \{1\} \times \widetilde{\text{TH}}_k(\tilde{G}). \end{aligned}$$

This shows that  $\widetilde{\text{TH}}_k(\tilde{G}) = \mathbf{cl}(\Omega_k(\tilde{G}))$ .  $\square$

#### 4. MORE DISCUSSIONS ON ASSUMPTION 3.1.4

As we have seen, if Assumption 3.1.4 is satisfied, then we can obtain a hierarchy of nested semidefinite relaxations converging to  $\mathbf{cl}(\mathbf{co}(S))$ . In this section, we give more discussions on cases where Assumption 3.1.4 does not hold.

**4.1. Closedness at  $\infty$  of  $S$ .** We have mentioned that a semialgebraic set is closed at  $\infty$  in general [6]. Unfortunately, as we show below, the closedness condition does not hold on certain kinds of semialgebraic sets.

Let  $U$  be a semialgebraic set defined as

$$(4.1) \quad U := \left\{ x \in \mathbb{R}^n \mid \begin{array}{l} g_i(x) = 0, \ g_j(x) \geq 0, \ i = 1, \dots, m_1, \\ \deg(g_j) \text{ is even}, \ j = 1, \dots, m_2 \end{array} \right\}.$$

Denote

$$\begin{aligned} \tilde{U}^o &= \{ \tilde{x} \in \mathbb{R}^{n+1} \mid \tilde{g}_i(\tilde{x}) = 0, \ \tilde{g}_j(\tilde{x}) \geq 0, \ x_0 > 0, \ i = 1, \dots, m_1, \ j = 1, \dots, m_2 \}, \\ \tilde{U}^c &= \{ \tilde{x} \in \mathbb{R}^{n+1} \mid \tilde{g}_i(\tilde{x}) = 0, \ \tilde{g}_j(\tilde{x}) \geq 0, \ x_0 \geq 0, \ i = 1, \dots, m_1, \ j = 1, \dots, m_2 \}. \end{aligned}$$

**Proposition 4.1.1.** *Suppose  $U$  is not compact. If  $\mathbf{co}\left(\mathbf{cl}\left(\tilde{U}^\circ\right)\right)$  is closed and pointed, then  $U$  is not closed at  $\infty$ .*

*Proof.* Since  $U$  is not compact, there is a sequence  $\{u^{(k)}\}_{k=1}^\infty \subseteq U$  satisfying  $\lim_{k \rightarrow \infty} \|u^{(k)}\|_2 = \infty$ . Because  $\{(1, u^{(k)}) / \|(1, u^{(k)})\|_2\} \subseteq \tilde{U}^\circ$  is bounded, there exists a nonzero point  $\tilde{u} = (0, u_1, \dots, u_n) \in \mathbf{cl}\left(\tilde{U}^\circ\right)$ .

If  $\mathbf{co}\left(\mathbf{cl}\left(\tilde{U}^\circ\right)\right)$  is closed and pointed, by Theorem 2.1.6, we have  $-\tilde{u} \notin \mathbf{cl}\left(\tilde{U}^\circ\right)$ . However,  $\deg(g_j)$  is even for  $j = 1, \dots, m_2$ , it is straightforward to see both  $\tilde{u}$  and  $-\tilde{u}$  belong to  $\tilde{U}^c$ , which implies  $\mathbf{cl}\left(\tilde{U}^\circ\right) \neq \tilde{U}^c$ . Therefore,  $U$  is not closed at  $\infty$ .  $\square$

**Remark 4.1.2.** Consider the semialgebraic set  $S$  defined as in (3.1). Let  $\hat{g}_i$  be the homogeneous part of the highest degree of  $g_i$  for  $i = 1, \dots, m$  and  $D_S = \tilde{S}^c \setminus \mathbf{cl}\left(\tilde{S}^\circ\right)$ . If  $S$  is not closed at  $\infty$ , then

$$\emptyset \neq D_S \subseteq \{(0, x) \in \mathbb{R}^{n+1} \mid \hat{g}_1(x) \geq 0, \dots, \hat{g}_m(x) \geq 0\}.$$

Decompose  $D_S = D_S^1 \cup D_S^2$  where

$$D_S^1 = \left\{ (0, x) \in \mathbb{R}^{n+1} \mid (0, x) \in \tilde{S}^c \setminus \mathbf{cl}\left(\tilde{S}^\circ\right) \text{ but } (0, -x) \in \mathbf{cl}\left(\tilde{S}^\circ\right) \right\}$$

and

$$D_S^2 = \left\{ (0, x) \in \mathbb{R}^{n+1} \mid (0, x) \in \tilde{S}^c \setminus \mathbf{cl}\left(\tilde{S}^\circ\right) \text{ and } (0, -x) \notin \mathbf{cl}\left(\tilde{S}^\circ\right) \right\}.$$

If  $S$  is defined by (4.1), then by the proof of Proposition 4.1.1,  $D_S^1 \neq \emptyset$ . However, if  $\mathbf{co}\left(\mathbf{cl}\left(\tilde{S}^\circ\right)\right)$  is closed and pointed, there exists a linear function  $\tilde{l} \in \mathbb{P}[\tilde{X}]_1$  such that  $\tilde{l}(\tilde{x}) > 0$  on  $\mathbf{co}\left(\mathbf{cl}\left(\tilde{S}^\circ\right)\right) \setminus \{0\}$ . Adding the inequality  $\tilde{l}(\tilde{x}) \geq 0$  to the generators of  $\tilde{S}^\circ$ , or equivalently, adding  $\tilde{l}(1, x) \geq 0$  to the generators of  $S$ , it is clear that  $S, \tilde{S}^\circ$  remain the same but we have  $D_S^1 = \emptyset$ . As a result, the set  $S$  with new generators is more likely to be closed at  $\infty$ . This shows that the closedness at  $\infty$  depends not only on the geometry of  $S$ , but also on the generators of  $S$ .

**Example 3.1.8 (continued).** We have shown that the semialgebraic set  $S$  is not closed at  $\infty$ , which can also be verified by Proposition 4.1.1. According to Remark 4.1.2, we add  $x_0 + x_2 \geq 0$  to the generators of  $\tilde{S}^\circ$ , or equivalently, we add  $1 + x_2 \geq 0$  to the generators of  $S$ , then  $\tilde{S}^c \setminus \mathbf{cl}\left(\tilde{S}^\circ\right) = \emptyset$ . Therefore,  $S$  is closed at  $\infty$  with respect to its new generators. Since the geometry of  $S$  does not change,  $\mathbf{co}\left(\mathbf{cl}\left(\tilde{S}^\circ\right)\right)$  is still closed and pointed. The second order spectrahedron  $\Omega_2(\tilde{G})$  is shown (shaded) in Figure 7.  $\square$

**Example 4.1.3.** Consider the quartic bow curve

$$S := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^4 - x_1^2 x_2 + x_2^3 = 0\}$$

as shown (red) in Figure 8. We have

$$\tilde{S}^\circ = \{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_1^4 - x_0 x_1^2 x_2 + x_0 x_2^3 = 0, x_0 > 0\},$$

$$\tilde{S}^c = \{(x_0, x_1, x_2) \in \mathbb{R}^3 \mid x_1^4 - x_0 x_1^2 x_2 + x_0 x_2^3 = 0, x_0 \geq 0\}.$$

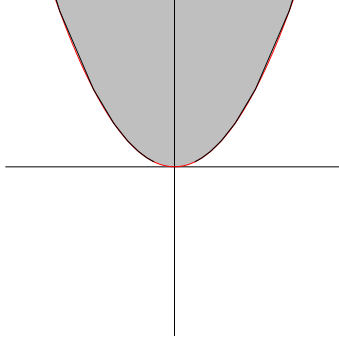


FIGURE 7. The parabola  $X_2 - X_1^2 = 0$  (red curve) and the second order spectrahedron  $\Omega_2(\tilde{G})$  (shaded) in Example 3.1.8.

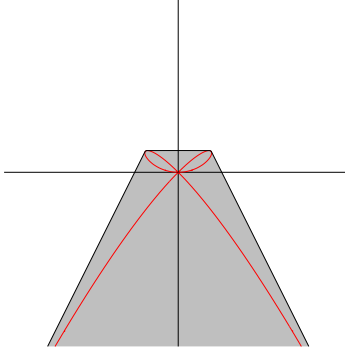


FIGURE 8. The bow curve  $S$  (red curve) and the third order spectrahedron  $\Omega_3(\tilde{G})$  (shaded) in Example 4.1.3.

We first show that  $\mathbf{co}\left(\mathbf{cl}\left(\tilde{S}^\circ\right)\right)$  is closed and pointed by proving the polynomial  $X_0 - X_2$  is positive on  $\mathbf{co}\left(\mathbf{cl}\left(\tilde{S}^\circ\right)\right) \setminus \{0\}$ . For every  $0 \neq \tilde{u} = (u_0, u_1, u_2) \in \mathbf{cl}\left(\tilde{S}^\circ\right)$ , we have  $u_1^4 - u_0 u_1^2 u_2 + u_0 u_2^3 = 0$ . If  $u_2 = 0$ , then  $u_1 = 0$  and  $u_0 - u_2 > 0$ . Assume  $u_2 \neq 0$ , then  $u_1^2 u_2 - u_2^3 \neq 0$ . Otherwise, we have  $u_1^2 = u_2^2$  and  $u_2^4 - u_0 u_2^3 + u_0 u_2^3 = 0$ , then  $u_2 = 0$ , a contradiction. Therefore,

$$\begin{aligned} u_0 - u_2 &= \frac{u_1^4}{u_1^2 u_2 - u_2^3} - u_2 \\ &= \frac{(u_1^2 - u_2^2)^2 + u_1^2 u_2^2}{u_1^2 u_2 - u_2^3} \\ &> 0, \end{aligned}$$

which implies  $\mathbf{co}\left(\mathbf{cl}\left(\tilde{S}^\circ\right)\right)$  is closed and pointed. Since  $S$  is of form (4.1), by Proposition 4.1.1,  $S$  is not closed at  $\infty$ . By Remark 4.1.2, we add  $1 - x_2 \geq 0$  to the generators of  $S$  to “force” it to be closed at  $\infty$ . The third order spectrahedron  $\Omega_3(\tilde{G})$  with the new generating set is shown (shaded) in Figure 8.  $\square$

**4.2. Pointedness of  $\text{co}(\text{cl}(\tilde{S}^\circ))$ .** When  $\text{co}(\text{cl}(\tilde{S}^\circ))$  is not pointed, we divide  $S$  into  $2^n$  parts along each axis. Let

$$(4.2) \quad \mathcal{E} := \{e = (e_1, \dots, e_n) \mid e_i \in \{0, 1\}, \ i = 1, \dots, n\}$$

and for each  $e \in \mathcal{E}$ ,

$$(4.3) \quad S_e := \{x \in \mathbb{R}^n \mid g_i(x) \geq 0, \ (-1)^{e_j} x_j \geq 0, \ i = 1, \dots, m, \ j = 1, \dots, n\}.$$

Then  $S = \bigcup_{e \in \mathcal{E}} S_e$  and  $|\mathcal{E}| = 2^n$ . For each  $e \in \mathcal{E}$ , define  $\tilde{S}_e^\circ$ ,  $\tilde{S}_e^c$ ,  $\tilde{S}_e$  and  $\tilde{G}_e$  as in (3.2) and (3.3). By Theorem 2.1.6, both of  $\text{co}(\text{cl}(\tilde{S}_e^\circ))$  and  $\text{co}(\tilde{S}_e^c)$  are closed and pointed for each  $e \in \mathcal{E}$ .

**Theorem 4.2.1.** *Let  $S \in \mathbb{R}^n$  be a semialgebraic set defined as in (3.1). Assume that*

1.  *$S$  is closed at  $\infty$ ;*
2. *For each  $e \in \mathcal{E}$ , PP-BDR property holds for  $\tilde{S}_e$ .*

*Then  $\text{cl}(\text{co}(S))$  is the closure of a projected spectrahedron.*

*Proof.* Fix an integer  $k'$  such that PP-BDR property holds for each  $\tilde{S}_e$  with order  $k'$ . Note that  $S_e$  may not be closed at  $\infty$  for some  $e \in \mathcal{E}$ . However, we show that

$$(4.4) \quad \text{cl}(\text{co}(S_e)) \subseteq \text{cl}(\Omega_{k'}(\tilde{G}_e)) \subseteq \text{cl}(\text{co}(S)), \quad \forall e \in \mathcal{E}.$$

By Theorem 3.2.1, we get

$$\text{cl}(\text{co}(S_e)) \subseteq \text{cl}(\Omega_{k'}(\tilde{G}_e)) \subseteq \widetilde{\text{TH}}_{k'}(\tilde{G}_e), \quad \forall e \in \mathcal{E}.$$

Fix a vector  $u \notin \text{cl}(\text{co}(S))$ . According to (3.7), there exists a polynomial  $\tilde{f} \in \mathbb{P}[\tilde{X}]_1$  such that  $\tilde{f}(1, u) < 0$  and  $\tilde{f}(\tilde{x}) \geq 0$  on  $\text{co}(\text{cl}(\tilde{S}^\circ))$ . Since  $\text{cl}(\tilde{S}^\circ) = \tilde{S}^c$  and  $\tilde{S}_e^c \subseteq \tilde{S}^c$  for each  $e \in \mathcal{E}$ , we have  $\tilde{f}(\tilde{x}) \geq 0$  on each  $\text{co}(\tilde{S}_e^c)$ . Because  $\text{co}(\tilde{S}_e^c)$  is closed and pointed, by Theorem 2.1.6, there exists a polynomial  $\tilde{g} \in \mathbb{P}[\tilde{X}]_1$  such that  $\tilde{g}(\tilde{x}) > 0$  on  $\text{co}(\tilde{S}_e^\circ)$ . We choose a small  $\epsilon > 0$  such that  $(\tilde{f} + \epsilon\tilde{g})(1, u) < 0$  and rename  $\tilde{f} + \epsilon\tilde{g}$  as  $\tilde{f}$ , then  $\tilde{f}(\tilde{x}) > 0$  on  $\tilde{S}_e^c$ . In particular,  $\tilde{f}(\tilde{x}) > 0$  on  $\tilde{S}_e$ . Since  $\tilde{S}_e$  satisfies PP-BDR property with order  $k'$ , we have  $\tilde{f} \in \mathcal{Q}_{k'}(\tilde{G})$  and  $u \notin \widetilde{\text{TH}}_{k'}(\tilde{G})$  due to the fact that  $\tilde{f}(1, u) < 0$ . It implies  $\widetilde{\text{TH}}_{k'}(\tilde{G}_e) \subseteq \text{cl}(\text{co}(S))$  and (4.4). Therefore, we have

$$\begin{aligned} \text{cl}(\text{co}(S)) &= \text{cl}\left(\text{co}\left(\bigcup_{e \in \mathcal{E}} S_e\right)\right) \\ &= \text{cl}\left(\text{co}\left(\bigcup_{e \in \mathcal{E}} \text{cl}(\text{co}(S_e))\right)\right) \\ &\subseteq \text{cl}\left(\text{co}\left(\bigcup_{e \in \mathcal{E}} \text{cl}(\Omega_{k'}(\tilde{G}_e))\right)\right) \\ &\subseteq \text{cl}(\text{co}(\text{cl}(\text{co}(S)))) \\ &= \text{cl}(\text{co}(S)), \end{aligned}$$

which implies

$$\mathbf{cl}(\mathbf{co}(S)) = \mathbf{cl}\left(\mathbf{co}\left(\bigcup_{e \in \mathcal{E}} \mathbf{cl}(\Omega_{k'}(\tilde{G}_e))\right)\right) = \mathbf{cl}\left(\mathbf{co}\left(\bigcup_{e \in \mathcal{E}} \Omega_{k'}(\tilde{G}_e)\right)\right).$$

Since each  $\Omega_{k'}(\tilde{G}_e)$  is a projected spectrahedron, by [7, Theorem 2.2], we have

$$\mathbf{cl}\left(\mathbf{co}\left(\bigcup_{e \in \mathcal{E}} \Omega_{k'}(\tilde{G}_e)\right)\right) = \mathbf{cl}\left(\sum_e \lambda_e x^{(e)} \mid \sum_e \lambda_e = 1, \lambda_e \geq 0, x^{(e)} \in \Omega_{k'}(\tilde{G}_e)\right)$$

which is the closure of a projected spectrahedron.  $\square$

**Remark 4.2.2.** If  $\mathcal{E}'$  is a subset of  $\mathcal{E}$  such that  $S = \bigcup_{e \in \mathcal{E}'} S_e$ , then according to the above proof, the conclusion of Theorem 4.2.1 still holds if we replace  $\mathcal{E}$  by  $\mathcal{E}'$ .

**Example 3.1.9 (continued).** For  $S = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2^3 - x_1^2 \geq 0\}$ , we have shown that  $\mathbf{co}(\mathbf{cl}(\tilde{S}^\circ))$  is not pointed, and the modified theta bodies (3.4) and Lasserre's relaxations (3.16) do not converge to  $\mathbf{co}(\mathbf{cl}(\tilde{S}))$ . Due to Remark 4.2.2, divide  $S$  into two parts

$$\begin{aligned} S_{(0,0)} &:= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2^3 - x_1^2 \geq 0, x_1 \geq 0, x_2 \geq 0\}, \\ S_{(1,0)} &:= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2^3 - x_1^2 \geq 0, -x_1 \geq 0, x_2 \geq 0\}. \end{aligned}$$

It is easy to check that PP-BDR property holds for  $\tilde{S}_{(0,0)}$  and  $\tilde{S}_{(1,0)}$  with order one. Thus for any  $k' \geq 1$ , we have

$$\begin{aligned} \Omega_{k'}(\tilde{G}_{(0,0)}) &= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0\}, \\ \Omega_{k'}(\tilde{G}_{(1,0)}) &= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \leq 0, x_2 \geq 0\}. \end{aligned}$$

Clearly,  $\mathbf{cl}(\mathbf{co}(\Omega_{k'}(\tilde{G}_{(0,0)}) \cup \Omega_{k'}(\tilde{G}_{(1,0)}))) = \mathbf{cl}(\mathbf{co}(S))$  for any integer  $k' \geq 1$ .  $\square$

However, if PP-BDR property does not hold with order  $k'$  for some  $\tilde{S}_e$ , according to the proof of Theorem 4.2.1,  $\mathbf{cl}(\Omega_{k'}(\tilde{G}_e))$  may not be a subset of  $\mathbf{cl}(\mathbf{co}(S))$  for some  $e \in \mathcal{E}$ . In this case,  $\mathbf{cl}(\mathbf{co}(\bigcup_{e \in \mathcal{E}} \Omega_{k'}(\tilde{G}_e)))$  may contain  $\mathbf{cl}(\mathbf{co}(S))$  strictly.

**Example 4.2.3.** Rotating the semialgebraic set  $S$  in Example 3.1.9 about the origin  $45^\circ$  counter-clockwise, we get

$$S' := \{(x_1, x_2) \in \mathbb{R}^2 \mid -\sqrt{2}(x_1 - x_2)^3 - 2(x_1 + x_2)^2 \geq 0\},$$

which is the left part of  $\mathbb{R}^2$  divided by the red curve in Figure 9. Then,  $\mathbf{cl}(\mathbf{co}(S))$  is the closed half space of  $\mathbb{R}^2$  partitioned by the line  $X_2 = X_1$ .

By Remark 4.2.2, set  $\mathcal{E}' = \{(0,0), (1,0), (1,1)\}$  and divide  $S' = \bigcup_{e \in \mathcal{E}'} S'_e$  defined as in (4.3). Clearly, for any integer  $k' \geq 1$ , we have

$$\Omega_{k'}(\tilde{G}'_{(1,0)}) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \leq 0, x_2 \geq 0\}.$$

The third order spectrahedral approximation  $\Omega_3(\tilde{G}'_{(0,0)})$  of  $S'_{(0,0)}$  is shown shaded in Figure 9. As we can see, the support line  $X_2 = X_1$  is approximated by  $X_2 = aX_1$  with  $a < 1$ . The same thing happens in the third quadrant. Numerically, we deduce  $\mathbf{cl}(\mathbf{co}(\bigcup_{e \in \mathcal{E}'} \Omega_3(\tilde{G}'_e))) = \mathbb{R}^2$  which contains  $\mathbf{cl}(\mathbf{co}(S))$  strictly.  $\square$

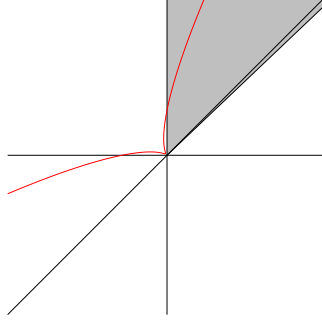


FIGURE 9. The third order spectrahedral approximation  $\Omega_3 \left( \tilde{G}'_{(0,0)} \right)$  (shaded) of  $S'_{(0,0)}$  in Example 4.2.3.

Therefore, when  $\text{co} \left( \text{cl} \left( \tilde{S}^\circ \right) \right)$  is not pointed, it becomes much more complicate to approximate  $\text{cl}(\text{co}(S))$  properly if PP-BDR property does not hold on  $\tilde{S}_e$  for some  $e \in \mathcal{E}$ . We leave this case for future investigations.

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#### APPENDIX A.

In the following, we give the proof of Proposition 2.1.5.

*Proof of Proposition 2.1.5.* We prove the properties are equivalent by showing the implications [2]

$$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (f) \Rightarrow (a).$$

- (a) $\Rightarrow$ (b). Since  $K$  is pointed, by (2.1), we have

$$\mathbf{cl}(K^* - K^*) = (K \cap -K)^* = 0^* = \mathbb{R}^n.$$

- (b) $\Rightarrow$ (c). It follows from the fact that  $K^* - K^*$  is a subspace in  $\mathbb{R}^n$  and therefore it is closed.
- (c) $\Rightarrow$ (d). Since  $K^*$  is always nonempty, by [17, Theorem 6.2], it has nonempty relative interior. By [17, Theorem 2.7] and (c), we have  $\mathbf{aff}(K^*) = K^* - K^* = \mathbb{R}^n$  and thus  $K^*$  has nonempty interior.
- (d) $\Rightarrow$ (e). Let  $y$  be an interior of  $K^*$ , then  $\langle y, x \rangle > 0$  for all  $0 \neq x \in K$ . Let  $\epsilon := \min\{\langle y, u \rangle \mid u \in K, \|u\|_2 = 1\}$ , then  $\epsilon > 0$  and  $\langle y, x \rangle \geq \epsilon\|x\|$  for all  $x \in K$ .
- (e) $\Rightarrow$ (f). By (e), there exist a vector  $y$  and real  $\epsilon > 0$  satisfying  $1/\|y\|_2 \leq \|x\|_2 \leq 1/\epsilon$  for all  $x \in C := \{x \in K \mid \langle x, y \rangle = 1\}$ , thus  $C$  is bounded and  $0 \notin \mathbf{cl}(C)$ . For every  $0 \neq u \in K$ , we have  $\langle u, y \rangle > 0$  and  $u/\langle u, y \rangle \in C$ , i.e.,  $C$  is a bounded base of  $K$ .
- (f) $\Rightarrow$ (a). Suppose  $C$  is a bounded base of  $K$ . If there exists  $0 \neq x \in K \cap -K$ , then  $c_1 x \in C$  and  $-c_2 x \in C$  for some  $c_1, c_2 \in \mathbb{R}_+$ . Since  $C$  is a convex set, we have  $0 \in C$ . This contradicts to the assumption that  $0 \notin \mathbf{cl}(C)$ .

□



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